1) a) $f(z) = \dfrac{\tan z}{z^2+z+1} = \dfrac{\sin z}{\cos z \cdot \underbrace{(z^2+z+1)}_{\left[z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i\right)\right]\cdot\left[z-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot i\right)\right]}}$

Isolated singular points: $-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\cdot i$

$-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\cdot i$

$\dfrac{\pi}{2}\cdot(2n+1)$, n integer

$$\operatorname*{Res}_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i} \frac{\sin z}{\cos z\cdot\left(z-\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i\right]\right)\cdot\left(z-\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot i\right]\right)}$$

$$= \frac{\sin\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i\right)}{\cos\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i\right)\cdot\left(\sqrt{3}\cdot i\right)}$$

$$= \frac{\sin\left(-\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)+i\cdot\cos\left(-\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)}{\left[\cos\left(-\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)-i\cdot\sin\left(-\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)\right]\cdot\sqrt{3}\cdot i}$$

$$= \frac{\sin\left(-\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)+i\cdot\cos\left(-\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}\cdot\underbrace{\left\{\sin\left(-\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)+i\cdot\cos\left(-\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)\right\}}}$$

$$= \frac{-\sin\left(\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)+i\cdot\cos\left(\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}\cdot\underbrace{\left\{-\sin\left(\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)+i\cdot\cos\left(\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)\right\}}}$$

$$\operatorname*{Res}_{z=-\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot i} \frac{\sin z}{\cos z \cdot \left(z-\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i\right]\right)\cdot\left(z-\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot i\right]\right)}$$

$$= \frac{\sin\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot i\right)}{\cos\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot i\right)\cdot -\sqrt{3}\cdot i}$$

$$= \frac{\sin\left(-\frac{1}{2}\right)\cdot\cosh\left(-\frac{\sqrt{3}}{2}\right)+i\cdot\cos\left(-\frac{1}{2}\right)\cdot\sinh\left(-\frac{\sqrt{3}}{2}\right)}{\left\{\cos\left(-\frac{1}{2}\right)\cdot\cosh\left(-\frac{\sqrt{3}}{2}\right)-i\cdot\sin\left(-\frac{1}{2}\right)\cdot\sinh\left(-\frac{\sqrt{3}}{2}\right)\right\}\cdot -\sqrt{3}\cdot i}$$

$$= \frac{-\sin\left(\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)-i\cdot\cos\left(\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}\cdot\left\{-\sin\left(\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)-i\cdot\cos\left(\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)\right\}}$$

$$= \frac{\sin\left(\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)+i\cdot\cos\left(\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}\cdot\left\{\sin\left(\frac{1}{2}\right)\cdot\sinh\left(\frac{\sqrt{3}}{2}\right)+i\cdot\cos\left(\frac{1}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\right)\right\}}$$

$$\operatorname*{Res}_{z=\frac{\pi}{2}\cdot(2n+1)} \frac{\overbrace{\sin z}^{p(z)}}{\underbrace{\cos z}_{q(z)}\cdot\left(z^2+z+1\right)} = \frac{\sin\left(\frac{\pi}{2}\cdot[2n+1]\right)}{\left[\frac{\pi^2}{4}\cdot(2n+1)^2+\frac{\pi}{2}\cdot(2n+1)+1\right]\cdot -\sin\left(\frac{\pi}{2}\cdot(2n+1)\right)}$$

$$= -\frac{1}{\frac{\pi^2}{4}\cdot(2n+1)^2+\frac{\pi}{2}\cdot(2n+1)+1}$$

b) $\frac{\sqrt{z}}{z^3-4z^2+4z} = \frac{\sqrt{z}}{z\cdot(z^2-4z+4)} = \frac{\sqrt{z}}{z\cdot(z-2)^2}$

Isolated singular points : $z = 2$

$z=0$ is a singular point but it is not an isolated singular point. So there is no residue at $z=0$.

$$\operatorname*{Res}_{z=2} \frac{\sqrt{z}}{z\cdot(z-2)^2} = \operatorname*{Res}_{z=2} \frac{1}{\sqrt{z}\cdot(z-2)^2}$$

$$= \left(\frac{1}{\sqrt{z}}\right)'\Bigg|_{z=2} = \left(z^{-\frac{1}{2}}\right)'\Bigg|_{z=2} = \left(-\frac{1}{2}\cdot z^{-\frac{3}{2}}\right)\Bigg|_{z=2}$$

$$= -\frac{1}{2}\cdot e^{-\frac{3}{2}\cdot \operatorname{Log} z}\Bigg|_{z=2} = -\frac{1}{2}\cdot e^{-\frac{3}{2}\cdot \ln 2}$$

$$= -\frac{1}{2}\cdot 2^{-\frac{3}{2}} = -\frac{1}{2}\cdot\frac{1}{2\sqrt{2}}$$

$$= -\frac{1}{4\sqrt{2}}$$

$$= -\frac{\sqrt{2}}{8}$$

c) $\dfrac{1}{10^z - e^z} = f(z)$

singular points of $f(z)$ satisfy $10^z = e^z$

$$10^z = e^{\ln 10 \cdot z} = e^z \qquad z = x + i \cdot y$$

$$e^{\ln 10 \cdot x} \cdot e^{i \cdot \ln 10 \cdot y} = e^x \cdot e^{i \cdot y} \cdot \underbrace{e^{i \cdot 2 \cdot n \cdot \pi}}_{1}$$

n = integer

$\ln 10 \cdot x = x \rightarrow x = 0$

$\ln 10 \cdot y = y + 2 \cdot n \cdot \pi$

$$y = \frac{2 \cdot n \cdot \pi}{\ln 10 - 1} \rightarrow z = i \cdot \frac{2 \cdot n \cdot \pi}{\ln 10 - 1}$$

$$f(z) = \frac{p(z)}{q(z)} = \frac{1}{10^z - e^z} \qquad p(z) = 1, \quad q(z) = 10^z - e^z$$

$$\operatorname*{Res}_{z = i \cdot \frac{2 \cdot n \cdot \pi}{\ln 10 - 1}} f(z) = \frac{1}{q'(z)\Big|_{z = i \cdot \frac{2 \cdot n \cdot \pi}{\ln 10 - 1}}}$$

$$q'(z) = \ln 10 \cdot 10^z - e^z$$

$$q'(z)\Big|_{z = i \cdot \frac{2 \cdot n \cdot \pi}{\ln 10 - 1}} = (\ln 10 - 1) \cdot e^{i \cdot \frac{2 \cdot n \cdot \pi}{\ln 10 - 1}}$$

$$\operatorname*{Res}_{z = i \cdot \frac{2 \cdot n \cdot \pi}{\ln 10 - 1}} f(z) = \frac{1}{(\ln 10 - 1) \cdot e^{i \cdot \frac{2 \cdot n \cdot \pi}{\ln 10 - 1}}}$$

d) $f(z) = \dfrac{\cos\left(\frac{1}{z}\right)}{\sin z}$

Isolated singular points : $z = n \cdot \pi$, n integer $n \neq 0$

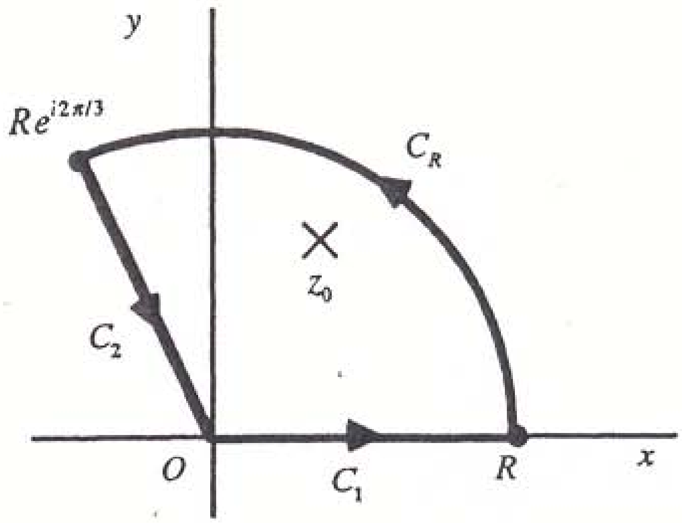
$z = 0$ is a singular point but it is not an isolated singular point so there is no residue at it.

$$\operatorname*{Res}_{\substack{z = n \cdot \pi \\ n \neq 0}} \frac{\cos\left(\frac{1}{z}\right) \to p(z)}{\sin z \to q(z)} = \frac{\cos\left(\frac{1}{n \cdot \pi}\right)}{\cos(n \cdot \pi)}$$

$$= \frac{\cos\left(\frac{1}{n \cdot \pi}\right)}{(-1)^n}$$

$$= (-1)^n \cdot \cos\left(\frac{1}{n \cdot \pi}\right)$$

2) The problem here is to establish the integration formula $\int_0^\infty \frac{dx}{(x^3+1)^2}$ using the simple closed contour shown below, where $R > 1$.



There is only one singularity of the function $f(z) = \frac{1}{(z^3+1)^2}$, $z_0 = e^{i\pi/3}$, that is interior to the closed contour when $R > 1$. According to the residue theorem,

$$\int_{C_1} \frac{dz}{(z^3+1)^2} + \int_{C_R} \frac{dz}{(z^3+1)^2} + \int_{C_2} \frac{dz}{(z^3+1)^2} = 2\pi i \operatorname*{Res}_{z=z_0} \frac{1}{(z^3+1)^2}$$

where the legs of the closed contour are as indicated in the figure. Since C_1 has parametric representation $z = r$ $(0 \le r \le R)$,

$$\int_{C_1} \frac{dz}{(z^3+1)^2} = \int_0^R \frac{dr}{(r^3+1)^2}$$

and, since $-C_2$ can be represented by $z = re^{i2\pi/3}$ $(0 \le r \le R)$,

$$\int_{C_2} \frac{dz}{(z^3+1)^2} = -\int_{-C_2} \frac{dz}{(z^3+1)^2} = -\int_0^R \frac{e^{i2\pi/3}\,dr}{((re^{i2\pi/3})^3+1)^2} = -e^{i2\pi/3}\int_0^R \frac{dr}{(r^3+1)^2}$$

Consequently,

$$(1-e^{i2\pi/3})\int_0^R \frac{dr}{(r^3+1)^2} = 2\pi i \operatorname*{Res}_{z=z_0} \frac{1}{(z^3+1)^2} - \int_{C_R} \frac{dz}{(z^3+1)^2}$$

$$\operatorname*{Res}_{z=z_0} \frac{1}{(z^3+1)^2} = -\frac{1}{9} - i \cdot \frac{\sqrt{3}}{9}$$

$$\left(\frac{3}{2} - i \cdot \frac{\sqrt{3}}{2}\right) \cdot \int_0^R \frac{dr}{(r^3+1)^2} = \frac{2\sqrt{3}\,\pi}{9} - i \cdot \frac{2\pi}{9} - \int_{C_R} \frac{dz}{(z^3+1)^2}$$

But

$$\left| \int_{C_R} \frac{dz}{(z^3+1)^2} \right| \leq \frac{1}{(R^3-1)^2} \cdot \frac{2\pi R}{3} \longrightarrow 0 \text{ as } R \to \infty.$$

So we get, as $R \to \infty$:

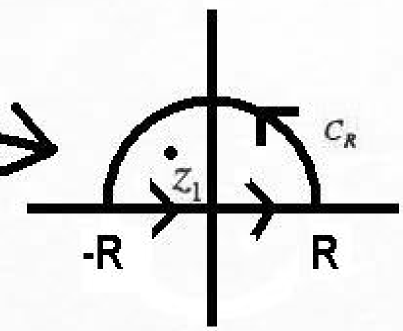
$$\int_0^\infty \frac{dr}{(r^3+1)^2} = \frac{\frac{2\pi}{9} \cdot (\sqrt{3} - i)}{\frac{3}{2} - i \cdot \frac{\sqrt{3}}{2}}$$

$$= \frac{4\sqrt{3}}{27} \cdot \pi$$

So;

$$\int_0^\infty \frac{dx}{(x^3+1)^2} = \frac{4\sqrt{3}}{27} \cdot \pi$$

3) To find the Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}dx$, we shall use the function $f(z) = \frac{z+1}{z^2+4z+5} = \frac{z+1}{(z-z_1)(z-\overline{z}_1)}$, where $\quad = -2+i$, and $\overline{z}_1 = -2-i$, and the same simple closed contour as in Exercise 9. In this case,



$$\int_{-R}^{R} \frac{(x+1)e^{ix}\,dx}{x^2+4x+5} + \int_{C_R} f(z)e^{iz}dz = 2\pi i B,$$

where

$$B = \operatorname*{Res}_{z=z_1}\left[\frac{(z+1)e^{iz}}{(z-z_1)(z-\overline{z}_1)}\right] = \frac{(z_1+1)e^{iz_1}}{(z-\overline{z}_1)} = \frac{(-1+i)e^{-2i}}{2ei}.$$

Thus

$$\int_{-R}^{R} \frac{(x+1)\cos x}{x^2+4x+5}dx = \operatorname{Re}(2\pi i B) - \int_{C_R} f(z)e^{iz},$$

or

$$\int_{-R}^{R} \frac{(x+1)\cos x}{x^2+4x+5}dx = \frac{\pi}{e}(\sin 2 - \cos 2) - \int_{C_R} f(z)e^{iz}dz.$$

Finally, we observe that if z is a point on C_R, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R+1}{(R-|z_1|)(R-|\overline{z}_1|)} = \frac{R+1}{(R-\sqrt{5})^2} \to 0 \text{ as } R \to \infty.$$

Limit (1), Sec. 74, then tells us that

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iz}dz\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right| \to 0 \text{ as } R \to \infty,$$

and so

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}dx = \frac{\pi}{e}(\sin 2 - \cos 2).$$

4) Let C be the positively oriented unit circle $|z|=1$. In view of the binomial formula (Sec. 3)

$$\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_C \left(\frac{z-z^{-1}}{2i}\right)^{2n} \frac{dz}{iz} = \frac{1}{2^{2n+1}(-1)^n i}\int_C \frac{(z-z^{-1})^{2n}}{z} dz$$

$$= \frac{1}{2^{2n+1}(-1)^n i}\int_C \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-k}(-z^{-1})^k z^{-1} dz$$

$$= \frac{1}{2^{2n+1}(-1)^n i}\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \int_C z^{2n-2k-1}\, dz.$$

Now each of these last integrals has value zero except when $k=n$:

$$\int_C z^{-1} dz = 2\pi i.$$

Consequently,

$$\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2^{2n+1}(-1)^n i} \cdot \frac{(2n)!(-1)^n 2\pi i}{(n!)^2} = \frac{(2n)!}{2^{2n}(n!)^2}\pi.$$

5) The problem here is to derive the integration formulas

$$I_1 = \int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}\,dx = \frac{\pi^2}{6} \quad \text{and} \quad I_2 = \int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{3}}$$

by integrating the function

$$f(z) = \frac{z^{1/3}\log z}{z^2+1} = \frac{e^{(1/3)\log z}\log z}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right),$$

around the contour shown in **Figure 1.**

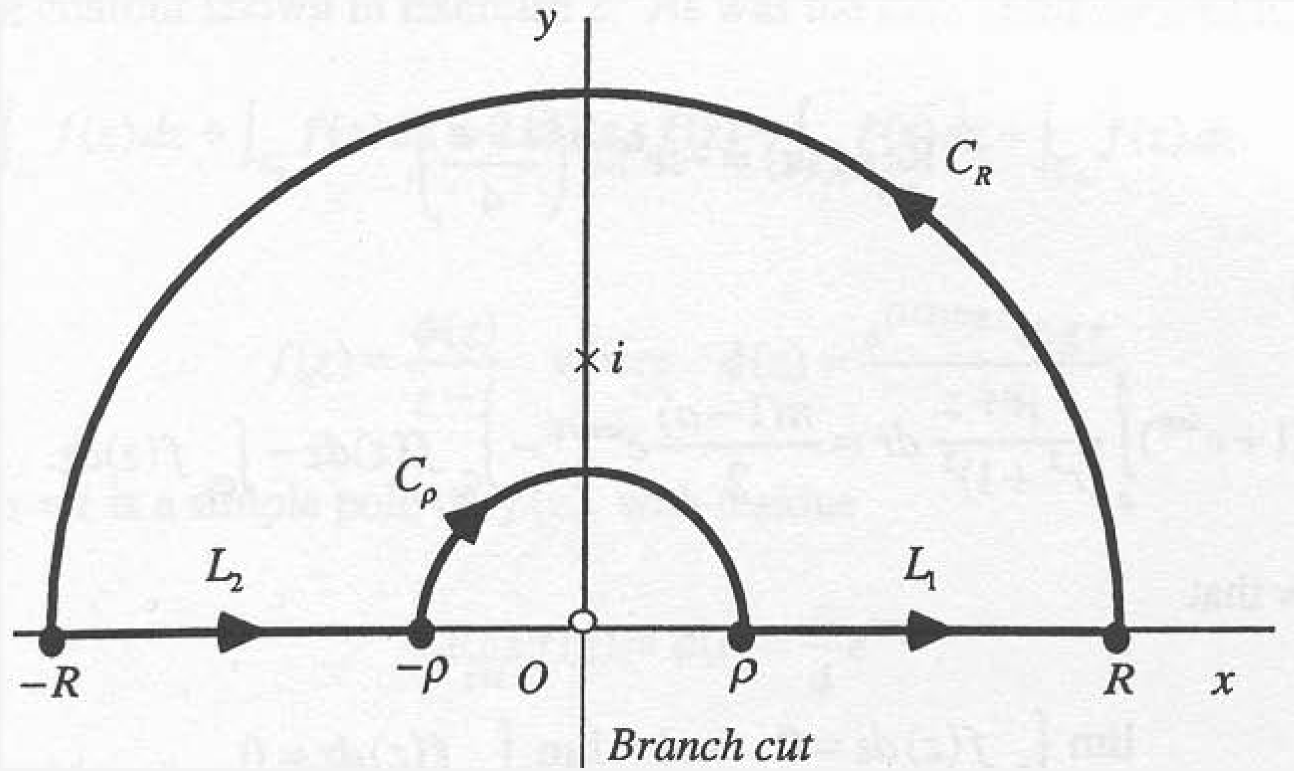


Figure 1

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)\,dz.$$

Since

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{e^{(1/3)\log z}\log z}{z+i},$$

the point $z = i$ is a simple pole of $f(z)$, with residue

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{\pi}{4}e^{i\pi/6}.$$

The parametric representations

$$L_1 : z = re^{i0} = r\ (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r\ (\rho \le r \le R)$$

can be used to write

$$\int_{L_1} f(z)dz = \int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr \quad \text{and} \quad \int_{L_2} f(z)dz = e^{i\pi/3}\int_\rho^R \frac{\sqrt[3]{r}\ln r + i\pi\sqrt[3]{r}}{r^2+1}dr.$$

Thus

$$\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr + e^{i\pi/3}\int_\rho^R \frac{\sqrt[3]{r}\ln r + i\pi\sqrt[3]{r}}{r^2+1}dr = \frac{\pi^2}{2}ie^{i\pi/6} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

By equating real parts on each side of this equation, we have

$$\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr + \cos(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr - \pi\sin(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}}{r^2+1}dr = -\frac{\pi^2}{2}\sin(\pi/6)$$
$$-\operatorname{Re}\int_{C_\rho} f(z)dz - \operatorname{Re}\int_{C_R} f(z)dz;$$

and equating imaginary parts yields

$$\sin(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr + \pi\cos(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}}{r^2+1}dr = \frac{\pi^2}{2}\cos(\pi/6)$$
$$-\operatorname{Im}\int_{C_\rho} f(z)dz - \operatorname{Im}\int_{C_R} f(z)dz.$$

Now $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, $\cos(\pi/3) = \frac{1}{2}$, $\sin(\pi/6) = \frac{1}{2}$, $\cos(\pi/6) = \frac{\sqrt{3}}{2}$ and it is routine to show that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

Thus

$$\frac{3}{2}\int_0^\infty \frac{\sqrt[3]{r}\ln r}{r^2+1}dr - \frac{\pi\sqrt{3}}{2}\int_0^\infty \frac{\sqrt[3]{r}}{r^2+1}dr = -\frac{\pi^2}{4},$$

$$\frac{\sqrt{3}}{2}\int_0^\infty \frac{\sqrt[3]{r}\ln r}{r^2+1}dr + \frac{\pi}{2}\int_0^\infty \frac{\sqrt[3]{r}}{r^2+1}dr = \frac{\pi^2\sqrt{3}}{4}.$$

That is,

$$\frac{3}{2}I_1 - \frac{\pi\sqrt{3}}{2}I_2 = -\frac{\pi^2}{4},$$

$$\frac{\sqrt{3}}{2}I_1 + \frac{\pi}{2}I_2 = \frac{\pi^2\sqrt{3}}{4}.$$

Solving these simultaneous equations for I_1 and I_2, we arrive at the desired integration formulas.