Math 206 - Homework #9

Solutions

1) $$y''(t) + y(t) = \sum_{k=0}^{\infty} \delta(t-k\pi) \quad , \quad y(0) = y'(0) = 0$$

$\mathcal{L} \downarrow$

$$s^2 Y(s) - s.y(0) - y'(0) + Y(s) = \sum_{k=0}^{\infty} e^{-sk\pi}$$

$$(s^2+1)\,Y(s) = \sum_{k=0}^{\infty} e^{-sk\pi}$$

$$Y(s) = \sum_{k=0}^{\infty} \frac{e^{-sk\pi}}{s^2+1}$$

$\mathcal{L}^{-1} \downarrow$

$$\therefore \quad y(t) = \sum_{k=0}^{\infty} \sin(t-k\pi).\,H(t-k\pi)$$

$$y'(t) = \sum_{k=0}^{\infty} \cos(t-k\pi).\,H(t-k\pi) + \underbrace{\sum_{k=0}^{\infty} \sin(t-k\pi).\,\delta(t-k\pi)}_{= \sum_{k=0}^{\infty} \sin(k\pi-k\pi).\delta(t-k\pi) = 0}$$

$$\therefore \quad y'(t) = \sum_{k=0}^{\infty} \cos(t-k\pi)\,H(t-k\pi)$$

$$y''(t) = \sum_{k=0}^{\infty} -\sin(t-k\pi)\,H(t-k\pi) + \underbrace{\sum_{k=0}^{\infty} \cos(t-k\pi)\,\delta(t-k\pi)}$$

$$= \sum_{k=0}^{\infty} \cos(k\pi-k\pi).\,\delta(t-k\pi)$$

$$= \sum_{k=0}^{\infty} \delta(t-k\pi)$$

$$\therefore \quad y''(t) = -\sum_{k=0}^{\infty} \sin(t-k\pi)\,H(t-k\pi) + \sum_{k=0}^{\infty} \delta(t-k\pi)$$

$$y''(t)+y(t)=\sum_{k=0}^{\infty}\sin(t-k\pi)H(t-k\pi)-\sum_{k=0}^{\infty}\sin(t-k\pi)H(t-k\pi)+\sum_{k=0}^{\infty}\delta(t-k\pi)$$

$$=\sum_{k=0}^{\infty}\delta(t-k\pi) \quad \checkmark$$

$$y(0)=\sum_{k=0}^{\infty}\underbrace{\sin(0-k\pi)}_{0}\underbrace{H(0-k\pi)}_{0}=0 \quad \checkmark$$

$$y'(0)=\sum_{k=0}^{\infty}\underbrace{\cos(0-k\pi)}_{(-1)^k}\underbrace{H(0-k\pi)}_{0}=0 \quad \checkmark$$

■ QED

2) $f(t)=H(t)+H(t-1)+H(t-2)-3H(t-4)$

$$\mathcal{L}\{f(t)\}=F(s)=\frac{1}{s}+\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-3\frac{e^{-4s}}{s}$$

$$y''(t)-3y'(t)+2y(t)=f(t) \qquad y(0)=y'(0)=0$$

$\mathcal{L}$ ↓

$$s^2Y(s)-\cancel{sy(0)}-\cancel{y'(0)}-3sY(s)+\cancel{3y(0)}+2Y(s)=F(s)$$

$$(s^2-3s+2)Y(s)=F(s)$$

$$Y(s)=\frac{F(s)}{s^2-3s+2}=\frac{F(s)}{(s-1)(s-2)}=F(s)\cdot\left[\frac{1}{s-2}-\frac{1}{s-1}\right]$$

$$Y(s)=\frac{1}{s(s-2)}+\frac{e^{-s}}{s(s-2)}+\frac{e^{-2s}}{s(s-2)}-3\frac{e^{-4s}}{s(s-2)}$$

$$-\frac{1}{s(s-1)}-\frac{e^{-s}}{s(s-1)}-\frac{e^{-2s}}{s(s-1)}+3\frac{e^{-4s}}{s(s-1)}$$

$$\frac{1}{s(s-2)} = \frac{1}{2}\left(\frac{1}{s-2} - \frac{1}{s}\right) \xrightarrow{\mathcal{L}^{-1}} \frac{1}{2}\left(e^{2t}-1\right) H(t)$$

$$\frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} \xrightarrow{\mathcal{L}^{-1}} (e^{t}-1)\, H(t)$$

$$y(t) = \frac{1}{2}\left[e^{2t}-1\right] H(t) + \frac{1}{2}\left[e^{2(t-1)}-1\right] H(t-1)$$

$$+ \frac{1}{2}\left[e^{2(t-2)}-1\right] H(t-2) - \frac{3}{2}\left[e^{2(t-4)}-1\right] H(t-4)$$

$$- \left[e^{t}-1\right] H(t) - \left[e^{(t-1)}-1\right] H(t-1) - \left[e^{(t-2)}-1\right] H(t-2) + 3\left[e^{(t-4)}-1\right] H(t-4)$$

3) $$x'' - 3x' + y' + 2x - y = 0 \qquad x(0) = 0, \quad x'(0) = 0$$

$$x' + y' - 2x + y = 0 \qquad y(0) = -1$$

$$s^2 X(s) \cancel{- s\,x(0)} \cancel{- x'(0)} - 3s\, X(s) + \cancel{3x(0)} + s\,Y(s) - \overset{-1}{\widetilde{y(0)}} + 2X(s) - Y(s) = 0$$

$$s\,X(s) \cancel{- x(0)} + s\,Y(s) - \underbrace{y(0)}_{-1} - 2X(s) + Y(s) = 0$$

$$X(s)\left(s^2 - 3s + 2\right) + Y(s)\,(s-1) = -1$$

$$X(s)\,(s-2) + Y(s)\,(s+1) = -1 \rightarrow X(s)\,(s-2) = -1 - Y(s)\,(s+1)$$

$$\underbrace{X(s)\,(s-2)}_{-1-Y(s)(s+1)}(s-1) + Y(s)\,(s-1) = -1$$

$$-(s-1) - Y(s)(s+1)(s-1) + Y(s)\,(s-1) = -1$$

$$Y(s) = -\frac{s-2}{s(s-1)} = \frac{-2}{s} + \frac{1}{s-1}$$

$$y(t) = -2H(t) + e^t H(t)$$

$$= H(t) \cdot \{e^t - 2\}$$

$$X(s)(s^2-3s+2) + \underbrace{Y(s)}_{-\frac{s-2}{s(s-1)}}(s-1) = -1$$

$$X(s)(s-1)(s-2) = -1 + \frac{s-2}{s} = \frac{-2}{s}$$

$$X(s) = \frac{-2}{s(s-1)(s-2)} = \frac{-1}{s} + \frac{2}{s-1} - \frac{1}{s-2}$$

$$x(t) = -H(t) + 2e^t H(t) - e^{2t} H(t)$$

$$= -H(t)\left(e^{2t} - 2e^t + 1\right)$$

$$= -H(t)\left(e^t - 1\right)^2$$

4) $$\frac{2s-5}{(s^2+9)^2} = \frac{2s}{(s^2+9)^2} - 5 \cdot \frac{1}{(s^2+9)^2}$$

$$\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{-6s}{(s^2+9)^2} \xrightarrow{\mathcal{L}^{-1}} -t\,\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) = -t.\sin 3t.$$

$$\frac{2s}{(s^2+9)^2} = -\frac{1}{3}\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{t}{3}\sin(3t).$$

$$\frac{d}{ds}\left(\frac{1}{s^2+9}\right) = \frac{-2s}{(s^2+9)^2}$$

$$-\frac{1}{(s^2+9)^2} = \frac{1}{2}\left\{\frac{1}{s}\left(\frac{d}{ds}\left[\frac{1}{s^2+9}\right]\right)\right\}$$

$$\mathcal{L}^{-1}\left\{-\frac{1}{(s^2+9)^2}\right\} = \frac{1}{2}\int_0^t f(u)\,du$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+9}\right)\right\} = -\frac{t}{3}\cdot\sin 3t$$

$$\mathcal{L}^{-1}\left\{\frac{-1}{(s^2+9)^2}\right\} = \frac{1}{6}\int_0^t -u\sin 3u\,du = -\frac{1}{6}\int_0^t u\sin 3u\,du$$

$$= \frac{1}{18}t\cos 3t - \frac{1}{54}\sin 3t$$

$$\mathcal{L}^{-1}\left(\frac{2s-5}{(s^2+9)^2}\right) = \mathcal{L}^{-1}\left(\frac{2s}{(s^2+9)^2}\right) + 5\mathcal{L}^{-1}\left(\frac{-1}{(s^2+9)^2}\right)$$

$$= \frac{t}{3}\sin(3t) + \frac{5}{18}t\cos(3t) - \frac{5}{54}\sin(3t)$$

5) $y'' + 2y' + y = \sin t \qquad y(0) = 3,\ y'(0) = 1$

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \frac{1}{s^2+1}$$

$$(s+1)^2\,Y(s) - 3s - 7 = \frac{1}{s^2+1}$$

$$Y(s) = \frac{3s+7}{(s+1)^2} + \frac{1}{(s^2+1)(s+1)^2}$$

$$= \frac{3s+7}{(s+1)^2} - \frac{s}{2(s^2+1)} + \frac{s+2}{2(s+1)^2}$$

$$= \frac{7s+16}{2(s+1)^2} - \frac{s}{2(s^2+1)}$$

$$Y(s) = \frac{7}{2(s+1)} + \frac{9}{2(s+1)^2} - \frac{s}{2(s^2+1)}$$

$$\mathcal{L}^{-1} \downarrow$$

$$y(t) = \frac{9}{2} t e^{-t} + \frac{7}{2} e^{-t} - \frac{1}{2} \cos t$$