Date: November 26, 2008, Wednesday NAME:..
Time: 17:30-19:30
Altıntaş & Sertöz STUDENT NO:..

Math 206 Complex Calculus – Midterm Exam II – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) Evaluate

$$\int_C \frac{\cot z}{z^2}\, dz$$

where C is the circle $|z| = 3$ described in the positive sense.

Solution: The only singularity of the integrand in the given domain is $z = 0$. If $\cot z = \cdots + a_1 z + \cdots$, then the residue of $\dfrac{\cot z}{z^2}$ at $z = 0$ is a_1. The easiest way to obtain a_1 is to divide the series of $\cos z$ by that of $\sin z$. We then find that

$$\cot z = z^{-1} - \frac{1}{3}z - \frac{1}{45}z^3 - \frac{2}{945}z^5 + \cdots$$

and this gives $a_1 = -1/3$. Hence the value of the integral is $-2\pi i/3$ from the residue theorem.

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Q-2) Let

$$f(a) = \int_C \frac{e^{az}}{1+e^z}\, dz$$

where C is the rectangular contour shown in the figure below and a is a complex variable.

(i): Calculate $f(1)$

(ii): Calculate $f(1/4)$.

C is the rectangle, traversed in the positive direction, with corners at the points 2π, $2\pi + 2\pi i$, $-2\pi + 2\pi i$ and -2π.

Solution: The only singularity of the integrand in the given region is $z = \pi i$. By the residue theorem $f(a) = 2\pi i e^{(a-1)\pi i}$. Hence

$$f(1) = 2\pi i, \quad f(1/4) = 2\pi i(-1/\sqrt{2} - i/\sqrt{2}) = \sqrt{2}\pi(1-i).$$

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Q-3) Calculate the integral $\displaystyle\int_{|z|=2} \frac{z^{18}}{z^{19}-1}\, dz.$

Solution: Let $g(z) = \dfrac{z^{18}}{z^{19}-1}$. There are 19 singularities of the integrand in the given region. By the residue theorem the value of the integral is $2\pi i$ times the sum of all these 19 residues. But the sum of all the residues is also equal to the residue at zero of $\dfrac{1}{z^2} g(1/z) = \dfrac{1}{z}\dfrac{1}{1-z^{19}}$, which is easily seen to be 1. Hence the value of the integral is $2\pi i$.

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Q-4) Represent the function $f(z) = \dfrac{z}{(z-1)(z-3)}$ by its Laurent series in the domain $0 < |z-1| < 2$.

Solution:

$$\begin{aligned}
\frac{z}{(z-1)(z-3)} &= \frac{(z-1)+1}{(z-1)((z-1)-2)} \\
&= -\frac{1}{2}\frac{(z-1)+1}{(z-1)(1-\frac{(z-1)}{2})} \\
&= -\frac{1}{2}\left(1+\frac{1}{z-1}\right)\left(1+\frac{z-1}{2}+\cdots+\frac{(z-1)^n}{2^n}+\cdots\right) \\
&= -\frac{1}{2}\left[\left(1+\frac{z-1}{2}+\cdots+\frac{(z-1)^n}{2^n}+\cdots\right)+\right. \\
&\quad \left.\left(\frac{1}{z-1}+\frac{1}{2}+\frac{z-1}{4}+\cdots+\frac{(z-1)^n}{2^{n+1}}+\cdots\right)\right] \\
&= -\frac{1}{2}\left[\frac{1}{z-1}+\frac{3}{2}+\frac{3(z-1)}{4}+\cdots+\frac{3(z-1)^n}{2^{n+1}}+\cdots\right] \\
&= -\frac{1}{2(z-1)}-\frac{3}{4}-\frac{3(z-1)}{8}-\cdots-\frac{3(z-1)^n}{2^{n+2}}+\cdots.
\end{aligned}$$

Please send comments to `sertoz@bilkent.edu.tr`
