Math 213 Advanced Calculus
Midterm Exam I
<u>Solutions</u>

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1) Find the lim inf and lim sup of the following sequence as $n \to \infty$.

$$x_n = \frac{7n^2 - 8n + 1}{13n^2 - 71n + 38}(1 + (-1)^n) + \frac{703 + 39n^2 + 72n^3}{451 - 40n^2 + 143n^3}(1 + (-1)^{n+1})$$

Solution 1: Let $A(n) = 2\frac{7n^2 - 8n + 1}{13n^2 - 71n + 38}$, and $B(n) = 2\frac{703 + 39n^2 + 72n^3}{451 - 40n^2 + 143n^3}$. Then

$$x_n = \begin{cases} A(n), & \text{if } n \text{ is even} \\ B(n), & \text{if } n \text{ is odd} \end{cases}$$

We see that $\{x_n\}$ has only two accumulation points, given by the limits of the convergent subsequences $A(n)$ and $B(n)$. Clearly $\lim_{n\to\infty} A(n) = 14/13$ and $\lim_{n\to\infty} B(n) = 144/143$. Hence $\liminf x_n = 144/143$ and $\limsup x_n = 14/13$.

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2) Let $f(x) = [\arctan(x)]^2 \ln(1 + x^2)$ for $x \in \mathbb{R}$. Define

$$G(t) = \int_0^t f(x - t)dx.$$

Calculate $G'(-\sqrt{3})$. Simplify your answer.

Solution 2: Use change of variables $u = x - t$ in the defining equation of $G(t)$ to change it to

$$G(t) = \int_{-t}^0 f(u)du.$$

Then $G'(t) = -f(-t)(-1) = f(-t)$. And $G'(-\sqrt{3}) = f(\sqrt{3}) = \ln(1 + 3)[\arctan\sqrt{3}]^2 = 2\ln 2(\pi/3)^2 = (2\pi^2 \ln 2)/9$

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3) Let $\phi(x) = \lim_{s \to 0^+} \dfrac{e^{-1/s^3}}{s^x}$ for $x \in [1, \infty]$. Draw the graph of ϕ.

Solution 3: Let $x = 3n + \alpha$, where n is a nonnegative integer and α is a real number with $0 \leq \alpha < 3$. Change variables with $t = 1/s$. When $s \to 0^+$, then $t \to \infty$. So $\phi(x) = \lim_{t \to \infty} \dfrac{e^{-t^3}}{1/t^x} = \lim_{t \to \infty} \dfrac{t^{3n+\alpha}}{e^{t^3}}$. Applying L'Hopital rule once we get $\phi(x) = \dfrac{(3n+\alpha)}{3} \lim_{t \to \infty} \dfrac{t^{3(n-1)+\alpha}}{e^{t^3}}$. Hence if we apply L'Hopital for $n+1$ times we should get $\phi(x) = \frac{(3n+\alpha)(3(n-1)+\alpha)\cdots(3+\alpha)(\alpha)}{3^{n+1}} \lim_{t\to\infty} \frac{1}{t^{3-\alpha}e^{t^3}} = 0$ for any $x = 3n + \alpha$. Hence ϕ is identically equal to 0 and drawing its graph should cause no problem!

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4) Prove that if f is integrable on $[0, 1]$ and $\beta > 0$, then

$$\lim_{n \to \infty} n^\alpha \int_0^{1/n^\beta} f(x)dx = 0, \text{ for all } \alpha < \beta.$$

Solution 4: This is from the book, page 117, exercise 3. Since f is integrable, it is bounded by definition, so let $M > 0$ be a constant such that $-M < f(x) < M$, for all $x \in [0, 1]$. For a fixed integer $n > 0$ let $P(n)$ be a partition of $[0, 1/n^\beta]$. Note that f is integrable on this subinterval. Then $(1/n^\beta)(-M) \leq L(f, P(n)) \leq \int_0^{1/n^\beta} f(x)dx \leq U(f, P(n)) < (1/n^\beta)(M)$. Here we used the fact that $|f(x)| < M$ and that the length of the interval $[0, 1/n^\beta]$ is $1/n^\beta$. It then follows that $-M/n^\beta \leq \int_0^{1/n^\beta} f(x)dx \leq M/n^\beta$. Multiplying all sides with the positive number n^α, for any $\alpha < \beta$ we get $-M/n^{\beta-\alpha} \leq (n^\alpha) \int_0^{1/n^\beta} f(x)dx \leq M/n^{\beta-\alpha}$. Note that $\lim_{n\to\infty} |M/n^{\beta-\alpha}| = 0$ since $\alpha < \beta$. Hence we get the required limit from the sandwich theorem.

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5) Suppose f is differentiable on $[0, 1]$ and that f' is one-to-one on $[0, 1]$. Assume that $f'(0) < f'(1)$. Show that f' is strictly increasing on $[0, 1]$.

Solution 5: This is from the book, page 98, exercise 7. First we claim that for any $c \in (0, 1)$, we must have $f'(0) < f'(c) < f'(1)$. Suppose to the contrary that there is a $c \in (0, 1)$ such that say $f'(c) < f'(0) < f'(1)$. Then by Duhamel's principle there is a point $d \in (c, 1)$ such that $f'(d) = f'(0)$ contradicting our condition that f' is one-to-one. Similarly assuming that $f'(0) < f'(1) < f'(c)$ leads to a contradiction. Now let x_1, x_2 be two points in $(0, 1)$ with $x_1 < x_2$. We want to show that $f'(x_1) < f'(x_2)$. Assume to the contrary that $f'(x_1) > f'(x_2)$. Note that equality is ruled out since f' is one-to-one. Also note that by the claim we proved above we must have $f'(x_1) > f'(x_2) > f'(0)$. Then again by Duhamel's principle there is an $x_0 \in (0, x_1)$ such that $f'(0) = f'(x_2)$, again contradicting the fact that f' is one-to-one. Hence f' is strictly increasing on $(0, 1)$.

Notes to the novice: You should have writen your answers in a readable and self explanatory manner. I do not try to read what you meant. I simply read what you wrote. If what you had in mind and what you wrote on paper are different, beware that I mark only what is on the paper. Intentions don't count! Moreover, if your method is correct, whatever that means, you do not automatically get partial credits. Your partials are determined on how close you get to the correct result, and let me be the judge of it!!
Thank you...