Math 213 Advanced Calculus
Midterm Exam III

December 9, 2000 — Ali Sinan Sertöz

1) Show that the convergence $\left(1+\dfrac{x}{k}\right)^k \to e^x$ as $k \to \infty$ is uniform on every compact subset of $\mathbb{R}$. Show however that this convergence is not uniform on $\mathbb{R}$.

Solution: Setting $f_k(x) = (1+x/k)^k$, observe that each f_k is pointwise monotone, continuous and the limiting function is also continuous. Then use Dini's theorem to conclude that the convergence is uniform on each compact subset. To show that the convergence cannot be uniform on all of $\mathbb{R}$ observe that $e^x - f_k(x)$ is an unbounded function of x on $\mathbb{R}$.

2) Calculate the following limits, if they exist:

(a) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x-y+z}{(x^2+y^2+z^2)^\alpha}, \quad \alpha < \frac{1}{2}.$ **(b)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}.$

Solution: The first one is in class notes. Here it helps if you observe that $|x-y+z| \le 3\max\{|x|,|y|,|z|\}$, and $(x^2+y^2+z^2)^\alpha \ge [\max\{|x|,|y|,|z|\}]^{2\alpha}$. Then you can show that the limit exists and is zero. For the second one if you use the path $y=\lambda x^3$, you find that the limit depends on the path, so no limit exists.

3) Let $E = \{\dfrac{1}{n} \in \mathbb{R} \mid n \in \mathbb{N}\ \}$. Construct an open cover for E which does not admit any finite subcover. Now extend your cover, by adding more open sets if necessary, to an open cover of $E \cup \{0\}$, and produce a finite subcover for $E \cup \{0\}$, and explain why you can always do this.

Solution: One such cover can be given by

$$U_n = \left(\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right), \frac{1}{n} + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right),$$

for $n = 1, 2, \ldots$. Since every element of E is contained in one and only one U_n, no finite subcover exists. However no matter which open set U you add to cover the point 0, the complement of the open set U will contain only finitely many elements of E and consequently a finite subcover will exist. This all follows from Heine-Borel theorem which says that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

4) Find the equation of the plane passing through the points $p_1 = (1, 1, 2)$, $p_2 = (3, 3, 4)$ and $p_3 = (5, 6, 8)$.

Solution: The vector $[p_2 - p_1] \times [p_3 - p_1]$ is orthogonal to the plane and if $p = (x, y, z)$ is an arbitrary point in the plane then the vector $[p - p_1]$ is orthogonal to the above cross-product. So if you dot product them you should get zero. This gives

$$\det \begin{vmatrix} x-1 & y-1 & z-2 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 2x - 4y + 2z - 2 = 0.$$

5) For every $n \in \mathbb{N}$ show that there is a function $f \in \mathcal{C}^n(\mathbb{R})$ such that $f^{(n+1)}(x)$ does not exist for any $x \in \mathbb{R}$.

Let u be an everywhere continuous but nowhere differentiable function on $\mathbb{R}$. Since u is continuous, by the fundamental theorem of calculus the function $f(x) = \displaystyle\int_0^x u(t)dt$ is differentiable with $f' = u$. Hence $f \in \mathcal{C}^1(\mathbb{R})$ but clearly f'' does not exist. Now successively integrating f we get the desired result.