Date: April 14, 2010, Wednesday NAME:..
Time: 08:40-10:30
Ali Sinan Sertöz STUDENT NO:..

Math 214 Advanced Calculus – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Selçuk Erdem

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Q-1) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function.

(a): Define what it means for f to be differentiable at $p \in \mathbb{R}^n$.

(b): Mark each of the following statements as TRUE or FALSE.

Grading: Each correct answer is 2 points, each wrong answer is -3 points. No answer is 0 points.

(i) If f is differentiable at p, then f is continuous at p.

(ii) If f is differentiable at p, then all first order partial derivatives of f exist at p.

(iii) If all first order partial derivatives of f exist at p, then f is differentiable at p.

(iv) If all second order partial derivatives of f exist at p, then f is differentiable at p.

(v) If f is differentiable at p, then all first order partial derivatives of f exist and are continuous at p.

(vi) If all directional derivatives of f exist at p, then f is continuous at p.

Solution:

(a): f is differentiable at p if there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(p+h) - f(p) - T(h)}{|h|} = 0,$$

where $h \in \mathbb{R}^n$.

(b) T-T-F-T-F-F

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Q-2) (a): State the implicit function theorem for a function $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ at a point $(a, b) \in \mathbb{R}^{n+m}$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

(b): Show that the system of equations

$$\begin{aligned} u^2 + v^2 + xyz &= 33 \\ \frac{u}{x} + \frac{v}{y} + \frac{4}{z^2} &= 7 \end{aligned}$$

can be solved for u and v in terms of x, y and z around the point $(x, y, z) = (2, 1, 2)$ such that $u(2, 1, 2) = 2$ and $v(2, 1, 2) = 5$.

Solution:

(a): Let $f = (f_1, \ldots, f_n)$ and let the coordinates of $\mathbb{R}^{n+m}$ be given by $(x_1, \ldots, x_n, y_1, \ldots, y_m)$. Assume that $f(a, b) = 0$. If $\det \left(\frac{\partial f_i}{\partial x_i}(a, b) \right)_{1 \le i,j \le n} \neq 0$, then around $b \in \mathbb{R}^m$ there exists a unique differentiable function g such that $g(b) = a$ and $f(g(t), t) = 0$ for all t in some neighbourhood of b in $\mathbb{R}^m$.

(b): Let $f : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by

$$f(u, v, x, y, z) = (f_1, f_2) = (u^2 + v^2 + xyz - 33,\ \frac{u}{x} + \frac{v}{y} + \frac{4}{z^2} - 7).$$

Then

$$\det \left(\frac{\partial f_i}{\partial x_i}(2, 5, 2, 1, 2) \right) = \det \begin{pmatrix} 4 & 10 \\ 1/2 & 1 \end{pmatrix} = -1 \neq 0.$$

So u and v can be solved in terms of x, y and z around $(2, 1, 2)$.

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Q-3) (a): State the inverse function theorem around the origin for a function $f:\mathbb{R}^n \to \mathbb{R}^n$.
(b): Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(x,y) = \left(1 + x + 3y + x^3 + y^4,\ \cos x + 2\sin y + \tan(x^2 + y^3)\right).$$

Show that f is invertible around $(0,0)$ and find $D(f^{-1})(1,1)$, where D denotes the total derivative.

Solution:

(a): If $\det Df(0) \neq 0$, then f is invertible on some neighbourhood of the origin. Moreover, $[Df^{-1}(f(0))] = [Df(0)]^{-1}$.

(b):

$$Df(0,0) = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}.$$

This matrix is invertible, so f is locally invertible and since $f(0,0) = (1,1)$, we have

$$D(f^{-1})(1,1) = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3/2 \\ 0 & 1/2 \end{pmatrix}.$$

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Q-4) Prove or disprove that the following function is differentiable at the origin.

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 1.$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0.$$

If f is differentiable at the origin, then its total derivative there must be $(1,0)$.

Now we check the definition of differentiability of f at the origin.

$$\frac{f(x,y) - f(0,0) - (1,0)\cdot(x,y)}{(x^2+y^2)^{1/2}} = \frac{-2xy^2}{(x^2+y^2)^{3/2}} = \phi(x,y).$$

Since $\phi(x,\lambda x) = \dfrac{-2\lambda^2}{(1+\lambda^2)^{3/2}}$, its limit as (x,y) goes to $(0,0)$ does not exist. Hence f is not differentiable at the origin.

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Q-5) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function.

(a): Let p be a point in $\mathbb{R}^n$ and u be a unit vector in $\mathbb{R}^n$. Define the directional derivative of f in the direction of the vector u at the point p.

(b): Prove or disprove that if f is differentiable at p, then its directional derivatives in all directions exist at p.

Solution:

(a): $D_u f(p) := \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t}$ when the derivative exists.

(b): Assume that f is differentiable at p. Take $h = tu$. By definition we have

$$0 = \lim_{t\to 0} \left| \frac{f(p+tu) - f(p) - T(tu)}{t} \right| = \lim_{t\to 0} \left| \frac{f(p+tu) - f(p)}{t} - T(u) \right|.$$

Hence $D_u f(p)$ exists and is equal to $T(u)$.