Solutions for Math 302 Homework-2 prepared by Ersin Üreyen.

Question 1: Let $H = \{z \in \mathbb{C} \mid Im\ z \geq 0\ \}$ and $f : H \longrightarrow \mathbb{C}$ be a nonconstant analytic function with $\sup_{z \in H} |f(z)| = 1$. Construct such a function f with

(i) $|f(z_0)| = 1$ for some $z_0 \in H$.

(ii) $|f(z)| < 1$ for all $z \in H$.

Solution 1. (i) Let $f(z) = e^{iz}$. Then $|f(z)| = |e^{i(x+iy)}| = e^{Re(ix-y)} = e^{-y} \leq 1$, since if $z = x + iy \in H$, then $y \geq 0$. For $z = x$, $x \in \mathbb{R}$, we have $|f(z)| = 1$.

(ii) Let $f(z) = z/(z + i)$. Then

$$|f(z)| = \sqrt{\frac{x^2 + y^2}{x^2 + (y+1)^2}} < 1,$$

since $y \geq 0$. To show that $\sup_{z \in H} |f(z)| = 1$, let $z_n = ni$, $n \in \mathbb{N}$. Then $|f(z_n)| = n/(n + 1) \to 1$ as $n \to \infty$.

Question 2: Let $D = \{z \in \mathbb{C} \mid Re\ z \geq 0\ \}$. Construct a nonconstant analytic function $f : D \longrightarrow \mathbb{C}$ with $|f(z)| \leq 1$ on D such that for every $\epsilon > 0$ there is a corresponding $A_\epsilon \in \mathbb{R}$ with $|f(z)| \leq A_\epsilon e^{\epsilon|z|}$ for all $z \in D$.

Solution 2. Let $f(z) = e^{-z}$. Then $|f(z)| = |e^{-z}| = e^{Re(-x-iy)} = e^{-x} \leq 1$, since if $z = x + iy \in D$, then $x \geq 0$.

Let $A_\epsilon = 1$. Since $e^{\epsilon|z|} \geq 1$, it follows that $|f(z)| \leq 1 \leq A_\epsilon e^{\epsilon|z|}$, $\forall z \in D$.

Question 3: Find a counterexample to Corollary 16.6 on page 202 when D is a proper subset of $\mathbb{C}$ but is not compact.

Solution 3. Let $A = \{z \in \mathbb{C} \mid Imz > 0\}$. Then $\partial A = \{z \in \mathbb{C} \mid Imz = 0\}$. Let $u_1(x, y) = 0$, $u_2(x, y) = y$. Clearly, u_1 and u_2 agree on ∂A but $u_1(z) \neq u_2(z)$, $z \in A$.

Question 4: Consider the function $g(z)$ constructed in the proof of Theorem 16.8 on page 205. Show that

(i) g is continuous in the unit disk.

(ii) g is analytic in the unit disk.

Solution 4. Let

$$f(\theta, z) = \frac{e^{i\theta} + z}{e^{i\theta} - z}, \qquad \theta \in [0, 2\pi],\ z \in D(0; 1).$$

(i) Since u is continuous on $C(0;1)$ and $C(0;1)$ is compact, there exists M such that

$$|u(e^{i\theta})| \leq M, \qquad \theta \in [0, 2\pi].$$

I.way: Fix $z_0 \in D(0;1)$. Let $d = (1 - |z_0|)$. We have

$$|f(\theta, z) - f(\theta, z_0)| = \left| \frac{e^{i\theta} + z}{e^{i\theta} - z} - \frac{e^{i\theta} + z_0}{e^{i\theta} - z_0} \right| = \left| \frac{2(z - z_0)}{(e^{i\theta} - z)(e^{i\theta} - z_0)} \right|.$$

Take $\epsilon > 0$. Let

$$\delta = \min\{\frac{d}{2}, \frac{\epsilon d^2}{4M}\}.$$

By triangular inequality, $|e^{i\theta} - z_0| \geq |e^{i\theta}| - |z_0| = 1 - |z_0| = d$. Again, by triangular inequality, for $|z - z_0| < \delta$,

$$|e^{i\theta} - z| \geq |e^{i\theta} - z_0| - |z - z_0| \geq d - \frac{d}{2} = \frac{d}{2}.$$

Hence, for $|z - z_0| < \delta$ and for all $\theta \in [0, 2\pi]$,

$$|f(\theta, z) - f(\theta, z_0)| < \frac{2\epsilon d^2/(4M)}{d \cdot d/2} = \frac{\epsilon}{M}. \tag{1}$$

Therefore, for $|z - z_0| < \delta$,

$$\begin{aligned} |g(z) - g(z_0)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} u(e^{i\theta}) \left[f(\theta, z) - f(\theta, z_0) \right] d\theta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} M |f(\theta, z) - f(\theta, z_0)| d\theta \\ &\leq \frac{1}{2\pi} 2\pi M \frac{\epsilon}{M} = \epsilon. \end{aligned}$$

This shows that g is continuous at z_0. Since $z_0 \in D(0;1)$ is arbitrary, g is continuous on $D(0;1)$.

II.way. Fix $z_0 \in D$. Let $d = (1 - |z_0|)$. Let $E = \{z \mid |z - z_0| \leq d/2\}$ and $F = [0, 2\pi] \times E$. Since $f(\theta, z)$ is continuous on F and F is compact, f is uniformly continuous on F. Take $\epsilon > 0$. By uniform continuity of f, there exists $\delta < d/2$ such that

$$|f(\theta', z) - f(\theta, z_0)| < \frac{\epsilon}{M}, \qquad \text{for } |\theta' - \theta| < \delta \text{ and } |z - z_0| < \delta.$$

Take $\theta' = \theta$. Then

$$|f(\theta, z) - f(\theta, z_0)| < \frac{\epsilon}{M}, \qquad \text{for } |z - z_0| < \delta,\ \theta \in [0, 2\pi]. \tag{2}$$

This shows (1). Now proceed as above.

The crucial point in the above argument is to show that the δ in (2) is *independent* of θ. The following argument is not sufficient: "Since f is continuous with respect to z, there exists δ such that (2) holds."

(ii) Let Γ be the boundary of an arbitrary closed rectangle lying in $D(0;1)$. The function $u(e^{i\theta})f(\theta, z)$ is continuous on $[0, 2\pi] \times \Gamma$. By Fubini's theorem

$$\int_\Gamma g(z)dz = \frac{1}{2\pi}\int_\Gamma \int_0^{2\pi} u(e^{i\theta})f(\theta, z)d\theta dz = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}) \int_\Gamma f(\theta, z)dzd\theta.$$

Since $f(\theta, z)$, as a function of z, is analytic in $D(0;1)$, by Cauchy's theorem $\int_\Gamma f(\theta, z)dz = 0$. This shows that $\int_\Gamma g(z)dz = 0$. By Morera's theorem g is analytic in $D(0;1)$.

Question 5: Find a C-harmonic function $u(x, y)$ on the unit disk D with $u(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ on ∂D, where the $a, b, \ldots, f$ are arbitrary real constants.

Solution 5. Let $u_1(x, y) = (x^2 - y^2 + 1)/2$. Since $u_{xx} + u_{yy} = 0$, u_1 is harmonic in $\mathbb{C}$. On ∂D we have $x^2 + y^2 = 1$. Therefore,

$$u_1(x, y) = \frac{x^2 - y^2 + (x^2 + y^2)}{2} = x^2, \qquad (x, y) \in \partial D.$$

Let $u_2(x, y) = (y^2 - x^2 + 1)/2$. Similarly, u_2 is harmonic in $\mathbb{C}$, and on ∂D, $u_2(x, y) = y^2$.

Let $u(x, y) = au_1(x, y) + bxy + cu_2(x, y) + dx + ey + f$. Then u is harmonic in $\mathbb{C}$, therefore C-harmonic in D and satisfies the required conditions. Note that, by Corollary 16.6, u is unique.
