Date: December 29, 2006, Friday Ali Sinan Sertöz

Math 302 Complex Calculus – Make-up Exam – Solutions

Q-1) Use residue theory to evaluate $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$.

Solution: Consider the path from $-R$ to R along the real line and then along the semicircle centered at the origin with radius R and lying in the upper half plane, with $R > 2$. Integrate the function along this contour. On one hand the integral is $2\pi i$ times the sum of the residues at $z = i$ and $z = 2i$, on the other hand the limit as $R \to \infty$ is twice the above integral. This gives the value of the integral as $\pi/6$.

Q-2) Find a conformal mapping of the first quadrant onto the unit disk mapping the points $1+i$ and 0 onto the points 0 and i respectively.

Solution: Take z and send it to $w = z^2$ first so that the region becomes the upper half plane. Then using the general form of a conformal map from the upper half plane onto the unit disk we find that the required mapping is

$$z \mapsto -i\,\frac{z^2-2i}{z^2+2i},$$

see page 172.

Q-3) Find a C-harmonic function on the unit disc which restricts to y^4 on the boundary.

Solution: Using the real parts of z^4 and z^2 we find that the required function is $(1/8)\,x^4 - (3/4)\,x^2y^2 + (1/8)\,y^4 - (1/2)\,x^2 + (1/2)\,y^2 + (3/8)$.

Q-4) Let $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}t^{z-1}\,dz$ for $\Re z > 0$. Show that $\Gamma(z)$ can be extended as a meromorphic function to the entire complex plane with simple poles at non-positive integers.

Solution: See pages 235-236.

Q-5) Show that the sum $1 - \dfrac{1}{2^z} + \dfrac{1}{3^z} - \dfrac{1}{4^z} + \cdots + \dfrac{(-1)^{n+1}}{n^z} + \cdots$, which converges absolutely for $\Re z > 1$, represents an entire function.

Solution: This is Exercise 7 on page 242 with its solution on page 288.
