Date: October 11, 2006, Wednesday
Time: 10:40-12:30

NAME:..

STUDENT NO:..

Math 302 Complex Calculus – Midterm Exam I – Solutions

Q-1) Let R be the domain obtained by removing the non-negative real numbers from $\mathbb{C}$. Construct a conformal mapping from $\mathbb{C}$ onto R if such a map exists, and explain why if it does not exist.

Solution: Suppose $f : \mathbb{C} \to R$ be such a map. Let $\phi : R \to D$ be a conformal isomorphism between R and the unit disk U, whose existence is assured by the Riemann mapping theorem. Then $\phi \circ f : \mathbb{C} \to U$ is a nonconstant bounded entire map onto U, which must be constant. This contradiction shows that no such map exists.

Note that it is not enough to quote Riemann mapping theorem here since it guarantees the existence of conformal isomorphism between two simply connected proper subsets of $\mathbb{C}$ but does not explicitly say anything about $\mathbb{C}$.

Q-2) Let $T = \{z = x + iy \in \mathbb{C} \mid 0 < x < 1,\ y > 0\}$, and $L_c = \{z = x + iy \in \mathbb{C} \mid x = c,\ y > 0\}$ for $0 < c < 1$. Describe $f(L_c)$ and $f(T)$ where $f(z) = 1/(1 - z)$.

Solution: $f(z) = X + iY$ where $X(x, y) = \dfrac{1 - x}{(1 - x)^2 + y^2}$ and $Y(x, y) = \dfrac{y}{(1 - x)^2 + y^2}$. Note that $X(c, y)^2 + Y(c, y)^2 - \dfrac{1}{1 - c} X(c, y) = 0$ for all y, and the image is a circle. So $f(L_c)$ is the semicircle with radius $\dfrac{1}{2(1 - c)}$ and center at $(\dfrac{1}{2(1 - c)}, 0)$, with $Y > 0$. Using this we conclude that $f(T)$ is the first quadrant, $X, Y > 0$ with the semicircle $X^2 + Y^2 - X = 0,\ Y > 0$ removed.

Q-3) Evaluate $I_a = \displaystyle\int_0^\infty \frac{x^2}{(x^2 + a^2)^3}\, dx$, where $a > 0$.

Solution: Let $\phi(x) = \dfrac{x^2}{(x^2 + a^2)^3} = \dfrac{f(x)}{(x - ia)^3}$ where $f(x) = \dfrac{x^2}{(x + ia)^3}$. Residue of $\phi(x)$ at $x = ia$ is $f''(ai)/2 = -i/(16a^3)$. Then by using the usual semicircle path and taking the usual limits we find that $I_a = \dfrac{\pi}{16a^3}$.

Q-4) Show that $\displaystyle\sum_{n=0}^{\infty} \binom{3n}{2n} \frac{1}{8^n} = \sum_{j=1}^{k} Res_{z=z_j} f(z)$, for some function $f(z)$ and some points $z_1, \ldots, z_k$. Find explicitly the function $f(z)$ and the points $z_1, \ldots, z_k$. Justify your calculations.

Solution: Let C be the circle $|z| = 2$. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{3n}{2n} \frac{1}{8^n} &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{(1+z)^{3n}}{(8z^2)^n} \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \left[\frac{(1+z)^3}{(8z^2)} \right]^n \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_C f(z)\, dz
\end{aligned}$$

where $f(z) = \dfrac{8z}{8z^2 - (1+z)^3}$. Its poles inside C are $z_1 = 1$ and $z_2 = 2 - \sqrt{5}$. Sum of these residues gives the total sum as 2.34. Check that $\left| \dfrac{(1+z)^3}{8z^2} \right| \leq \dfrac{27}{32} < 1$ for $z \in C$, which justifies the interchange of infinite sum and the integral due to uniform convergence.
