Due Date: October 24, 2011 Monday NAME:..

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Math 302 Complex Calculus II – Homework – Solutions

1	2
10	10
10	10

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Apply the contour integral method we studies to the evaluation of the sum

$$\sum_{n=0}^{\infty} \frac{1}{n^2+n+1},$$

and write the answer in decimal expansion with at least 8 digits after the decimal point.

Solution:

The zeros of z^2+z+1 are $z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Our method shows that

$$\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} \frac{1}{n^2+n+1} = -\sum_{k=1}^{2} \text{Res}\left(\frac{\pi \cot \pi z}{z^2+z+1}, z_k\right) =: \alpha_0.$$

Now

$$\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} \frac{1}{n^2+n+1} = \sum_{-\infty<n\leq -1} \frac{1}{n^2+n+1} + \sum_{n=0}^{\infty} \frac{1}{n^2+n+1},$$

and the first sum on the right hand side becomes $\sum_{n=0}^{\infty} \frac{1}{n^2+n+1}$ when n is replaced by $-n-1$. Thus we have

$$\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} \frac{1}{n^2+n+1} = 2\sum_{n=0}^{\infty} \frac{1}{n^2+n+1} = \alpha_0$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n^2+n+1} = \frac{\alpha_0}{2} = 1.798147281....$$

Note: $\text{Res}\left(\frac{\pi \cot \pi z}{z^2+z+1}, z\right) = -\frac{\sqrt{3}\pi}{3} \tanh(\frac{\sqrt{3}\pi}{2})$ for $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}.$

NAME: STUDENT NO:

Q-2) Apply the contour integral method we studies to the evaluation of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 4n^3 + 6n^2 + 4n},$$

and write the answer in decimal expansion with at least 8 digits after the decimal point.

Solution:

For ease of notation we set $g(z) = z^4 + 4z^3 + 6z^2 + 4z$. Its zeros are $0, -2, -1+i, -1-i$. Our method gives

$$\sum_{\substack{n=-\infty \\ n\neq 0,-2,-1+i,-1-i}}^{\infty} \frac{1}{g(n)} = -\sum_{z=0,-2,-1+i,-1-i} \text{Res}\left(\frac{\pi \cot \pi z}{g(z)}, z\right) =: \alpha_0.$$

By replacing n by $-n-1$, we get

$$\sum_{\substack{n=-\infty \\ n\neq 0,-2,-1+i,-1-i}}^{\infty} \frac{1}{g(n)} = \sum_{\substack{n=-\infty \\ n\neq \pm 1}}^{\infty} \frac{1}{n^4 - 1} = -1 + 2\sum_{n=2}^{\infty} \frac{1}{n^4-1} = \alpha_0.$$

Hence

$$\sum_{n=2}^{\infty} \frac{1}{n^4-1} = \frac{\alpha_0 + 1}{2} = 0.086662976....$$

Note:

$$\text{Res}\left(\frac{\pi \cot \pi z}{z^4 + 4z^3 + 6z^2 + 4z}, z\right) = -\frac{3}{8} \quad \text{for} \quad z = 0, -2,$$

$$\text{Res}\left(\frac{\pi \cot \pi z}{z^4 + 4z^3 + 6z^2 + 4z}, z\right) = \frac{\pi}{2}\coth(\pi z) \quad \text{for} \quad z = -1 \pm i,$$

and

$$\alpha_0 = -0.826674048...$$