Due Date: October 31, 2011 Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Calculus II – Homework – Solutions

3	4
10	10
10	10

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-3) Classify all the automorphisms of the first quadrant

Solution:

Let $f(z) = z^2$ with inverse $f^{-1}(z) = \sqrt{z}$ where we use the principal branch for the square root function. Then f is a conformal isomorphism of the first quadrant with the upper half plane and any automorphism of the first quadrant is of the form $f^{-1} \circ \phi \circ f$ where ϕ is an automorphism of the upper half plane.

Any automorphism of the upper half plane is of the form

$$\phi(z) = \frac{az+b}{cz+d} \quad \text{where } a, b, c, d \text{ are real and} \quad ad - bc > 0.$$

Hence any automorphism of the first quadrant is of the form

$$(f^{-1} \circ \phi \circ f)(z) = \sqrt{\frac{az^2+b}{cz^2+d}} \quad \text{where } a, b, c, d \text{ are real and} \quad ad - bc > 0.$$

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Q-4) This exercise aims to complete the proof of a theorem we did in class.

Fix $\alpha \in \mathbb{C}$ with $|\alpha| < 1$. Define

$$h(z) = \left(\frac{z-i}{z+i}\right)^{-1} \circ \left(\frac{z-\alpha}{1-\bar{\alpha}z}\right) \circ \left(\frac{z-i}{z+i}\right).$$

Show that

$$h(z) = \frac{az+b}{cz+d} \quad \text{with } a,b,c,d \in \mathbb{R} \text{ and } ad-bc > 0.$$

Solution:

Composition of Mobius transformations corresponds to multiplication of their coefficient matrices.

Let $\dfrac{z-i}{z+i}$ be represented by the matrix

$$A = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix},$$

and $\dfrac{z-\alpha}{1-\bar{\alpha}z}$ be represented by the matrix

$$B = \begin{bmatrix} 1 & -x-iy \\ -x+iy & 1 \end{bmatrix},$$

where $\alpha = x+iy$ with $x^2+y^2<1$.

Then $\left(\dfrac{z-i}{z+i}\right)^{-1} \circ \left(\dfrac{z-\alpha}{1-\bar{\alpha}z}\right) \circ \left(\dfrac{z-i}{z+i}\right)$ is represented by the matrix

$$C = \begin{bmatrix} 1-x & y \\ y & 1+x \end{bmatrix},$$

where $\det C = 1-x^2-y^2 > 0$ as claimed.