Due Date: July 4, 2011 Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Analysis II – Homework 6 – Solutions

1	2	TOTAL
10	10	20

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Show that $f(z) = e^z - z$ has infinitely many zeros and that each zero is simple. *(5+5 points)*

Solution:

Since $|f(z)| \leq e^{|z|^2}$ for large z, it is of finite order. If it has finitely many zeros, then it is of the form $Q(z)e^{P(z)}$ where $Q(z)$ and $P(z)$ are polynomials. We then have

$$e^{z-P(z)} - ze^{-P(z)} = Q(z).$$

But this is absurd since the LHS is clearly not a polynomial. Hence f must have infinitely many zeros. Let α be a root of $f(z) = 0$. $f'(\alpha) = e^{\alpha} - 1 = 0$ holds only when $\alpha = 0$ but 0 is not a root itself. So all roots must be simple.

Another way to see that the above equality is absurd is to check by induction that the n-th derivative of the LHS is of the form

$$e^{-P(z)} \left(R(z)e^z + S(z)\right)$$

where $R(z)$ and $S(z)$ are polynomials. When n is larger than the degree of $Q(z)$, this expression must be identically equal to zero. This forces e^z to be a rational function which is a contradiction. (A non-constant rational function has zeros and poles whereas e^z has none!)

NAME: STUDENT NO:

Q-2) Find explicitly a polynomial $P(x, y)$ such that it is harmonic in the unit disc D around the origin and restricts to x^3y^3 on the boundary of D. (Note that x^3y^3 is not harmonic anywhere except the origin.)
Show your work in detail.

Solution:

The boundary ∂D of D is given by $x^2 + y^2 = 1$. Here by putting $y^2 = 1 - x^2$ we get

$$P(x, y)|_{\partial D} = (x^3y^3)|_{\partial D} = x^3y - x^5y.$$

Let $u_1(x, y)$ be the harmonic extension of x^3y to D and $u_2(x, y)$ be the harmonic extension of x^5y. Then $P(x, y) = u_1(x, y) - u_2(x, y)$.

We start by searching for $u_2(x, y)$. Let $u(x, y) = \operatorname{Im} z^6 = 6x^5y-20x^3y^3+6xy^5$. On ∂D, $u(x, y)|_{\partial D} = 32x^5y - 32x^3y + 6xy$. We notice that xy is already harmonic everywhere and x^3y is the restriction of $u_1(x, y)$ to ∂D. This give

$$u_2(x, y)|_{\partial D} = x^5y = \frac{1}{32}u(x, y)|_{\partial D} + u_1(x, y)|_{\partial D} - \frac{3}{16}xy,$$

which in turn gives

$$u_2(x, y) = \frac{1}{32}u(x, y) + u_1(x, y) - \frac{3}{16}xy.$$

We now have

$$\begin{aligned} P(x, y) &= u_1(x, y) - u_2(x, y) \\ &= \frac{3}{16}xy - \frac{1}{32}u(x, y) \\ &= \frac{3}{16}xy - \frac{3}{16}x^5y + \frac{5}{8}x^3y^3 - \frac{3}{16}xy^5 \end{aligned}$$

which is harmonic and becomes $x^3y - x^5y$ when we put $y^2 = 1 - x^2$.