Date: 15 July 2011, Friday
Time: 13:40-15:40
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Midterm Exam 2 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer with little or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let D be the right half plane and f an analytic function on D with a continuous extension to $\bar{D}$. Assume that $|f(z)| \leq 1$ for $z \in \partial D$, and $|f(z)| \leq 2$ for $z \in D$. Give a detailed proof that actually $|f(z)| \leq 1$ for all $z \in D$.

Solution:

Fix a $z_0 \in D$. We want to show that $|f(z_0)| \leq 1$.

Fix a positive integer N. Consider the function $h_N(z) = \dfrac{f^N(z)}{z+1}$, $z \in D$. Clearly $|h_N(z)| \leq 1$ for $z = yi \in \partial D$.

Choose $R > |z_0|$. For all $z \in D$ with $|z| = R$, we must have $|h_N(z)| \leq \dfrac{2^N}{R}$.

Let D_R be the set of points in $\bar{D}$ with $|z| \leq R$. By the usual maximal modulus principle, for all $z \in D_R$, $|h_N(z)| \leq \max\left(\dfrac{2^N}{R}, 1\right)$. Choosing R large enough, we conclude that $|h_N(z_0)| \leq 1$.

This gives $|f(x_0)| \leq |z_0 + 1|^{1/N}$. Since $|z_0 + 1|^{1/N} \downarrow 1$ as $N \to \infty$, $|f(z_0)|$ cannot be equal to any value strictly larger than 1. i.e. $|f(x_0)| \leq 1$.

NAME: STUDENT NO:

Q-2) Let D be the unit disc around the origin. Find explicitly a function $f(x, y)$ such that f is harmonic on D and $f|_{\partial D} = x^3$. Can we arrange it so that $f(0, 0) = \pi$? Explain why or how.

Solution:

Let $u(x, y) = \operatorname{Re} z^3 = x^3 - 3xy^2$. Being the real part of an analytic function, $u(x, y)$ is harmonic on D.

On the unit circle we have $u|_{\partial D} = 4x^3 - 3x$. Observe that $3x$ is already harmonic everywhere. This gives $x^3 = \frac{1}{4}(u(x, y) + 3x)$ on the unit circle.

Thus we found a function $f(x, y) = \frac{1}{4}(u(x, y) + 3x) = \frac{1}{4}(x^3 - 3xy^2 + 3x)$ which is harmonic on D and restricts to x^3 on the boundary. Such a harmonic function is unique. Since we already have $f(0, 0) = 0$, there is no way to arrange it to get another value.

NAME: STUDENT NO:

Q-3) (i): Let f be an entire function with $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Prove or disprove that $f(z) = f(0)$ for all $z \in \mathbb{C}$.

(ii): Let f be an entire function with $\operatorname{Re} f(z) \leq 5$ for all $z \in \mathbb{C}$. Prove or disprove that $f(z) = f(0)$ for all $z \in \mathbb{C}$.

Solution:

(i): The function $h(z) = \dfrac{1}{f(z)}$ is entire since $f(z)$ is never zero and $|h(z)| \leq 1$ since $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. By Liouville's theorem, $h(z)$ is constant, and hence $f(z)$ is constant.

(ii): Since $\operatorname{Re} f \leq 5$, it follows that $\operatorname{Re}(6 - f(z)) \geq 1$ and then $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Now by the first part $6 - f$ and hence f is constant.

NAME: STUDENT NO:

Q-4) Show that $\prod_{k=0}^{\infty}\left(1+z^{2^k}\right)$ converges uniformly to a function f on some region D in $\mathbb{C}$. Find explicitly f and D.

Solution:

Let $P_n(z)=\prod_{k=0}^{n}\left(1+z^{2^k}\right)$. By induction we see that

$$P_n(z)=1+z+\cdots+z^{2^{n+1}-1}.$$

Then clearly $P_n(z)$ converges uniformly, since it is part of a geometric series, to $f(z)=\dfrac{1}{1-z}$ on the unit disc.

NAME: STUDENT NO:

Q-5) (i): Find the radius of convergence of $f(z) = \sum_{n=0}^{\infty} z^{n!}$ and prove or disprove that f has a natural boundary.

(ii): Recall that $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt$ for $\operatorname{Re} z > 0$. Choose any two real numbers $0 < x_0 < x_1$. Show that $|\Gamma(z)| \leq \Gamma(x_0) + \Gamma(x_1)$ for all z with $x_0 \leq \operatorname{Re} z \leq x_1$.

Solution:

(i): Setting $a_n = z^{n!}$ and using the ratio test we find

$$\left|\frac{a_{n+1}}{a_n}\right| = |z|^{n\,n!} < 1 \Rightarrow |z| < 1,$$

so the radius of convergence is 1.

For any positive integer N, let ω_N denote an N-th root of unity. For all $n \geq N$, we have $(\omega_N)^{n!} = \left[(\omega_N)^N\right]^{n!/N} = 1$, so the infinite series diverges at $z = \omega_N$. Since all the N-th roots of unity, for all N, are dense on the unit circle, $|z| = 1$ gives a natural boundary for the series.

(ii): The proof is easy and can be given by the following self explanatory sequence of inequalities.

$$\begin{aligned}
|\Gamma(z)| &= \left|\int_0^{\infty} e^{-t} t^{z-1}\, dt\right| \\
&\leq \int_0^{\infty} e^{-t} t^{x-1}\, dt \\
&= \int_0^{1} e^{-t} t^{x-1}\, dt + \int_1^{\infty} e^{-t} t^{x-1}\, dt \\
&\leq \int_0^{1} e^{-t} t^{x_0-1}\, dt + \int_1^{\infty} e^{-t} t^{x_1-1}\, dt \\
&\leq \Gamma(x_0) + \Gamma(x_1).
\end{aligned}$$