Date: 6 January 2014, Monday
Time: 15:30-18:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

In this exam you are allowed to use two A4 size cheat-sheets provided that they are written by yourself, no photocopies are allowed. Your name must be written on both of them during the exam. **You are asked to hand in your cheat-sheets together with your answers.**

NAME: STUDENT NO:

Q-1) Let L be a line in the complex plane and let T be a Mobius transformation sending L again to a line. Classify all such T.

Solution: This is Homework 2, Question 1.

There are two cases. Either $T(\infty) = \infty$, or $T(\infty) \in \mathbb{C}$.

If $T(\infty) = \infty$, then clearly T is linear.

If, on the other hand, $T(\infty) = w_0 \in \mathbb{C}$, then there must exist a $z_0 \in L \subset \mathbb{C}$ such that $T(z_0) = \infty$. Then we must have

$$T(z) = w_0 + \frac{w}{z - z_0},$$

for some $w \in \mathbb{C}$. Let $z_1 \in \mathbb{C}$ be a point on L other than z_0, and let $T(z_1) = w_1 \in T(L)$. Then

$$T(z) = w_0 + \frac{(z_1 - z_0)(w_1 - w_0)}{z - z_0}.$$

Any point on L is of the form $z_0 + t(z_1 - z_0)$ where $t \in \mathbb{R}$. Check that

$$T(z_0 + t(z_1 - z_0)) = w_0 + \frac{1}{t}(w_1 - w_0).$$

So the image is the line through w_0 and w_1.

Conclusion:

If $T(\infty) = \infty$, then let z_0 and z_1 be two different points on L. Let $w_0 = T(z_0)$ and $w_1 = T(z_1)$. Then

$$T(z) = \frac{w_1 - w_0}{z_1 - z_0}\,(z - z_0) + w_0.$$

If $T(\infty) = w_0 \in \mathbb{C}$, then let $z_0 \in L$ be such that $T(z_0) = \infty$. Choose any point $z_1 \in \mathbb{C}$ on L different than z_0 and set $w_1 = T(z_1)$. Then

$$T(z) = w_0 + \frac{(z_1 - z_0)(w_1 - w_0)}{z - z_0}.$$

Another solution which I learned from your papers is the following.

Any Mobius transformation is of the form $T(z) = \frac{az+b}{cz+d}$. If $c = 0$, then T is linear and sends L to a line. If $c \neq 0$, then $T(z) = (f_3 \circ f_2 \circ f_1)(z)$ where

$$f_1(z) = cz + d, \quad f_2(z) = \frac{1}{z}, \quad f_3(z) = \frac{a}{c} - \left(\frac{ad - bc}{c}\right) z.$$

Here f_1 and f_3 are linear and take lines to lines. But f_2 takes only those lines through the origin to lines. If we want T to send L to a line, then there must exist a point $z \in \mathbb{C}$ on L such that $cz + d = 0$.

Conclusion: The Mobius transformations sending L to a line are of the following form.

$$\{\frac{az + b}{cz + d} \mid ad - bc \neq 0 \text{ and } -\frac{d}{c} \in L\}.$$

Note that when $c = 0$, we can interpret $-d/c$ as infinity which is certainly on L and hence this description also covers the linear transformations.

NAME: STUDENT NO:

Q-2) Let f be an entire function of finite order with finitely many zeros. Show that either $f(z)$ is a polynomial or $f(z) + z$ has infinitely many zeros.

Solution: This is Homework 4, Question 1.

If $f(z)$ is a polynomial, then we are done. If $f(z)$ is not a polynomial, then we know that $f(z) = P(z)e^{Q(z)}$ where P and Q are polynomials and $Q(z)$ is not constant. Suppose that $g(z) = f(z) + z$ has finitely many zeros. Since g is entire and is of finite order, it must be of the form

$$g(z) = R(z)e^{S(z)},$$

where R and S are polynomials. This gives the equality

$$z + P(z)e^{Q(z)} = R(z)e^{S(z)}. \tag{$*$}$$

Taking the second derivatives of both sides and rearranging we obtain an equality of the form

$$P_0(z)e^{Q(z)} = R_0(z)e^{S(z)},$$

where $P_0(z)$ and $R_0(z)$ are polynomials. This gives

$$e^{Q(z)-S(z)} = \frac{R_0(z)}{P_0(z)}.$$

Since the LHS has neither zeros nor poles, the RHS being a rational function of z must be constant. This implies in particular that $S(z) = Q(z) + c_0$, where $c_0 \in \mathbb{C}$ is a constant. Putting this into equation $(*)$, we get

$$e^{Q(z)} = \frac{z}{R(z)e^{c_0} - P(z)}.$$

A similar argument as above forces $Q(z)$ to be a constant, which is a contradiction.

Hence $f(z) + z$ must have infinitely many zeros.

NAME: STUDENT NO:

Q-3) Does there exist a function $f(z)$ defined on $|z| < 1$ and analytic there with the property that the zero set of f consists of the set $\{1 - \frac{1}{k} \mid k = 1, 2, \ldots\}$? If yes, construct such a function, if no explain why?

Solution: This is an Exercise on Chapter 17.

First construct, using Weierstrass method, an entire function $g(z)$ which vanishes only at $z = 1, 2, \ldots$. For example such a function is $g(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{z/k}$.

Then $f(z) = g(1/(1-z))$ is the required function since $1/(1-z)$ is well defined for $|z| < 1$ and $1/(1-(1-1/k)) = k$.

I learned from Alper İncecik and Muhammed Said Gündoğan that $f(z) = \sin \frac{\pi}{1-z}$ does the trick.

NAME: STUDENT NO:

Q-4) Show that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges.

Solution: This is Homework 4, Question 4.

It is known that if $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} |z_k|^2$ converges, then $\prod_{k=1}^{\infty}(1+z_k)$ converges.

We have

$$\zeta(z) \prod_{p \text{ prime}} \left(1 - \frac{1}{p^z}\right) = 1, \ \operatorname{Re} z > 1.$$

Since $\lim_{z \to 1} \zeta(z) = \infty$, we must have $\prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)$ diverge to zero. Since $\sum 1/p^2$ converges, we must have $\sum 1/p$ diverge, which follows from the above remark.

NAME: STUDENT NO:

Q-5) Show that for any real number α, we have

$$\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)\, n^{\alpha}} = 1.$$

Hint: You may use the asymptotic formula $\Gamma(z) = \sqrt{(2\pi/z)}\,(z/e)^z\,(1+\epsilon_z)$ *for* $\operatorname{Re} z > 0$, *where* $\lim_{z\to\infty} \epsilon_z = 0$.
Remark: If you only solve the problem where α *is any positive integer, then you will get 12 points.*

Solution:

Using the hint we get

$$\begin{aligned} \frac{\Gamma(n+\alpha)}{\Gamma(n)\, n^{\alpha}} &= \frac{\frac{\sqrt{2\pi}}{\sqrt{n+\alpha}}\,\frac{(n+\alpha)^{n+\alpha}}{e^{n+\alpha}}\,(1+\epsilon_{n+\alpha})}{\frac{\sqrt{2\pi}}{\sqrt{n}}\,\frac{n^n}{e^n}\,(1+\epsilon_n)\, n^{\alpha}} \\ &= \sqrt{\frac{n}{n+\alpha}}\,\left(1+\frac{\alpha}{n}\right)^{n+\alpha} e^{-\alpha}\,\frac{1+\epsilon_{n+\alpha}}{1+\epsilon_n} \\ &\to 1 \text{ as } n\to\infty. \end{aligned}$$

If $\alpha = m$ is a positive integer, then

$$\frac{\Gamma(n+m)}{\Gamma(n)\, n^m} = \frac{(n-1)!\,\overbrace{n(n+1)\cdots(n+m-1)}^{m \text{ terms}}}{(n-1)!\, n^m} \to 1 \text{ as } n\to\infty.$$