Due Date: 28 April, Thursday 2016
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:.......................................

Math 302 Complex Analysis II – Homework 4 – Solutions

1	2	3	4	5	TOTAL
30	30	40	0	0	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework Assignments

(1) You may discuss the problems only with your classmates or with me. In particular you may not ask your assigned questions or any related question to online forums.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) It is absolutely mandatory that you write your answers alone. Any similarity with your written words and any other solution or any other source that I happen to know is a direct violation of honesty.

(4) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

NAME: STUDENT NO:

Q-1) Show that the unit circle is a natural boundary for the power series $\sum_{n=0}^{\infty} z^{n!}$, in two different ways; first show that the sum approaches to infinity as z approaches to a boundary point. Then use theorem 18.5 to show that the exponents grow sufficiently fast.

Solution:

If α is a k-th root of unity then for any $n \geq k$ we have $\alpha^{n!} = 1$. Hence as z approaches α, the power series becomes an infinite sum of 1s, hence infinite. Since such α are dense on the boundary, no point on the boundary has an open neighborhood where the power series is finite.

On the other hand, using theorem 18.5, we see that $\frac{(n+1)!}{n!} \to \infty$ as $n \to \infty$. Since the limit is strictly larger than 1, the theorem concludes that he circle of convergence is a natural boundary.

NAME: STUDENT NO:

Q-2) Show that $\cos \pi z = \prod_{k=0}^{\infty} \left(1 - \frac{4z^2}{(2k+1)^2}\right)$.

Solution:

Use the product formula for $\sin \pi z$ and note that $\sin 2\pi z = 2 \sin \pi z \cos \pi z$.

NAME: STUDENT NO:

Q-3) Show that $\sum_{p:\text{prime}} \frac{1}{p}$ diverges.

Solution:

This is exercise 8 on page 242 and the solution is on page 288.

Here is the solution I gave back in 2011:

We have the identity

$$\zeta(z) = \frac{1}{\prod_{p:prime}\left(1 - \frac{1}{p^z}\right)} \quad \text{for} \quad \operatorname{Re} z > 1.$$

We also know that: If $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} |z_k|^2$ converge, then $\prod_{k=1}^{\infty}(1 + z_k)$ converges. (This is an exercise from the book, Chapter 17, Exercise 3.)

Since $\zeta(z)$ becomes infinite as z approaches 1, the infinite product $\prod_{p:prime}\left(1 - \frac{1}{p^z}\right)$ diverges to zero.

Take z_k as the reciprocal of the k-th prime.

Since the infinite product diverges and $\sum |z_k|^2$ converges, we must have $\sum z_k$ diverge according to the above fact.

This proves that $\sum_{p\ prime} \frac{1}{p}$ diverges.