Date: 14 April 2016, Thursday
Time: 08:40-10:30
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Midterm 2 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

You may use the following formulas directly if you find them correct and meaningful.

$$\csc z = \frac{1}{z} + \frac{1}{6}\,z + \frac{7}{360}\,z^3 + \frac{31}{15120}\,z^5 + \frac{127}{604800}\,z^7 + \frac{73}{3421440}\,z^9 + \cdots$$

$$\psi(z) = \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k} \quad \text{is an entire function.}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right) = \gamma = 0.5772156649...$$

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Q-1) Evaluate the integral $\displaystyle\int_{|z|=1} \frac{\csc(z^2)}{z^{10}+8z^7}\,dz$.

Solution:

We first observe that

$$\frac{\csc z^2}{z^{10}+8z^7} = \frac{\csc z^2}{z^7}\,f(z),$$

where $f(z) = \dfrac{1}{z^3+8}$. Note that $f(z)$ is analytic in the unit disc since its poles have modulus 2. Hence the only singularity of the integrand in the unit disc is $z=0$.

Using the power series expansion of $\csc z$ given on the cover page, we find that

$$\frac{\csc z^2}{z^7} = \frac{1}{z^9} + \frac{1}{6z^5} + \frac{7}{360z} + \frac{31}{15120}z^3 + \cdots$$

Moreover we notice that the Taylor expansion of $f(z)$ around $z=0$ contains only terms of the form z^{3n}; in particular there is no term of the form z^4 or z^8. Hence

$$\mathrm{Res}\Big(\frac{\csc z^2}{z^{10}+8z^7}, z=0\Big) = f(0)\ \mathrm{Res}\Big(\frac{\csc z^2}{z^7}, z=0\Big) = \frac{1}{8}\cdot\frac{7}{360} = \frac{7}{2880}.$$

Therefore we have

$$\int_{|z|=1} \frac{\csc z^2}{z^{10}+8z^7}\,dz = (2\pi i)\frac{7}{2880} = \frac{7\pi i}{1440}.$$

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Q-2) Let $A_n = \{a_1, \ldots, a_n\}$ and $B_m = \{b_1, \ldots, b_m\}$ be two subsets of $\mathbb{C}$ of cardinalities n and m respectively, for $n, m > 0$. Set $A_0 = B_0 = \emptyset$.

(a) Suppose that there is a Mobius transformation ϕ such that $\phi : \mathbb{C} \setminus A_n \to \mathbb{C} \setminus B_m$ is an isomorphism. Show that $m = n$.

(b) Suppose that $n = m$. Can we find a Mobius transformation ϕ which sends $\mathbb{C} \setminus A_n$ isomorphically onto $\mathbb{C} \setminus B_n$?

Solution:

Part (a):
We have two cases:
Case 1: $\phi(\infty) = \infty$.
In this case we must have $\phi(A_n) = B_m$, and since ϕ is injective we must have $n = m$.

Case 2: $\phi(\infty) \in \mathbb{C}$.
Since ϕ is injective, no point of $\mathbb{C}$ will be mapped to $\phi(\infty)$, so $\phi(\infty) \in B_m$. Say $\phi(\infty) = b_m$. Since ϕ is onto $\mathbb{C} \cup \{\infty\}$, there must be a point in $\mathbb{C}$ which maps to ∞. Since $\mathbb{C} \setminus A_n$ is mapped into $\mathbb{C}$, this point must be in A_n. Say $\phi(a_n) = \infty$. We now have $\phi(A_n \setminus \{ a_n\}) = B_m \setminus \{b_m\}$. Since ϕ is injective, we must have $n = m$.

Part (b):
Case 1: $n = 0$. In this case $\phi(z) = z$ is the required isomorphism.

Case 2: $n = 1$. In this case $\phi(z) = z - a_1 + b_1$ is the required isomorphism.

Case 3: $n = 2$. In this case $\phi(z) = \left(\dfrac{b_1 - b_2}{a_1 - a_2}\right) z + \left(\dfrac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\right)$ is the required isomorphism.

Case 4: $n \geq 3$.
If we require that $\phi(\infty) = \infty$, then ϕ must be linear and is totally determined by a_1, a_2, b_1, b_2 as in Case 3 above. Then $\phi : \mathbb{C} \setminus A_n \to \mathbb{C} \setminus B_n$ is an isomorphism if and only if $b_j = \phi(a_j)$ for $j = 3, \ldots, n$. So the existence depends not only on the cardinality of B_n but also on its elements.

If on the other hand we require that $\phi(\infty) \in \mathbb{C}$, then arguing as we did above in Case 2 of Part (a), we will have $\phi(\infty) = b_n$ and $\phi(a_n) = \infty$. Then we are asking if $\phi(A_n \setminus \{a_n\}) = B_n \setminus \{b_n\}$. But a Mobius transformation is determined by its action on three points. We already specified its value on two points so we have margin for one more arbitrary assignment. We can set $\phi(a_1) = b_1$. Then $\phi : \mathbb{C} \setminus A_n \to \mathbb{C} \setminus B_n$ is an isomorphism if and only if $b_j = \phi(a_j)$ for $j = 2, \ldots, n-1$. When $n \geq 3$, this already gives at least one more condition so in general it will not be satisfied. Again we conclude that the existence of an isomorphism as aked in the question depends not only on the cardinality of B_n but also on its elements.

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Q-3) Find a function $f(x,y)$ such that f is continuous for $x^2+y^2 \leq 1$ and harmonic for $x^2+y^2<1$, satisfying $f(x,y)=x^2y$ when $x^2+y^2=1$.

Solution:

Let $\phi(x,y)=\mathrm{Im}(x+iy)^3=3x^2y-y^3$. On the unit disc we have $\phi(x,y)=4x^2y-y$, so we define

$$f(x,y)=\frac{\phi(x,y)+y}{4}=\frac{1}{4}(3x^2y-y^3+y).$$

Check that this f satisfies the requirements.

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Q-4) Show that $\lim_{n\to\infty} n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)$ converges for every $z\in\mathbb{C}$ and in fact defines an entire function.

Solution:

Using the information on the cover page we realize that $e^{-z/k}$ is a convergence factor for the resulting infinite product. We thus have

$$n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)=n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}e^{z/k}=e^{z(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n)}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}.$$

Taking limits as $n\to\infty$ and using again the information on the cover page we conclude that

$$\lim_{n\to\infty} n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)=e^{\gamma z}\psi(z)$$

and that it is an entire function.