Math 430/505 Complex Geometry – Assignments

Fall 2010, Ali Sinan Sertöz — Due Date: 25 November 2010 Thursday

1) Let $e_1,\ e_2$ be the standard real basis of $\mathbb{C}$. Show that the usual almost complex structure on $\mathbb{C}$ is given by the endomorphism $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let $A = \begin{pmatrix} 4 & 17 \\ -1 & -4 \end{pmatrix}$ be an endomorphism on $\mathbb{R}^2$. Check that A defines an almost complex structure on $\mathbb{R}^2$. Find a basis $f_1,\ f_2$ of $\mathbb{R}^2$ such that with respect to this basis, the matrix of the endomorphism A is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible endomorphism of $\mathbb{R}^2$. Find necessary and sufficient conditions on a, b, c, d such that A is an almost complex structure on $\mathbb{R}^2$. Find a basis u, v of $\mathbb{R}^2$ such that the matrix of A with respect to this basis is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

2) Show that any $w \in V^{1,0}$ is of the form $w = u - iJ(u)$ for some $u \in V$ where J is the almost complex structure on V.

3) Let M be a complex manifold and N a complex submanifold of dimensions m and n respectively. Show that for every $x \in N$ there exists an open set $U \subset M$ and a coordinate chart $z_1, \ldots, z_m$ on U for M such that

$$N \cap U = \{p \in U \mid z_{n+1}(p) = \cdots = z_m(p) = 0\}.$$

4) Let V be a finite dimensional real vector space with an almost complex structure J, and let $\phi : V \longrightarrow V$ be an automorphism commuting with the almost complex structure. Show that $\det \phi > 0$. Show that an almost complex structure defines a canonical orientation on V. Use this to show that a complex manifold is naturally oriented.

5) Let V be a real vector space of dimension 2 with a fixed orientation. Let $\langle\ ,\ \rangle$ be positive definite symmetric bilinear form on V. Construct an almost complex structure J on V compatible with the given scalar product. Conversely, let an almost complex structure J be given. Construct a positive definite symmetric bilinear form $\langle\ ,\ \rangle$ on V with which J is compatible. Speculate what may happen in higher dimensions.

6) Let $\alpha \in \bigwedge^k V_{\mathbb{C}}^*$ and $v_1, \ldots, v_k \in V_{\mathbb{C}}$. Prove the statement

$$\mathbb{I}(\alpha)(v_1, \ldots, v_k) = \alpha(\mathbb{I}(v_1), \ldots, \mathbb{I}(v_k)).$$

(see class notes for the notation.) Then use this to show explicitly that the associated fundamental form ω, of a finite dimensional vector space V with a positive definite symmetric bilinear form on it and a compatible almost complex structure J, is in $\bigwedge^{1,1} V^*$.

7) Prove the chain rule in the complex setting. Let $z = x + iy$ and $w = u + iv$. Let $f = f(w)$ and $w = g(z)$. Show that

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial z}$$

where as usual $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$ etc.

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