Due Date: 10 February 2015, Tuesday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:...

STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Homework 1 – Solutions

1	2	3	4	5	TOTAL
20	40	40	0	0	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework and Take-Home Exams

(1) You may discuss the problems only with your classmates or with me. In particular you may not ask your assigned questions or any related question to online forums.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) It is absolutely mandatory that you write your answers alone. Any similarity with your written words with any other solution or any other source that I happen to know is a direct violation of honesty.

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends.

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

(6) If I sense any hint of violation of any of the above rules in your paper I will either assume that you did not answer that question or that your contribution is only on the level of a scribe in which case I will assign a small fraction of the grade for that question.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work. Moreover I declare that every solution I wrote reflects my true understanding of the problem, and any sources used, ideas from friends or other sources included, are explicitly cited without exception. I am aware that violation of the above codes results not only in losing some meaningless grades but also of respect and dignity.

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Q-1) Let $\epsilon = (\epsilon_1, \epsilon_2)$ with $\epsilon_1, \epsilon_2 > 0$. Let $B_\epsilon(0)$ be the polydisc around the origin defined by

$$B_\epsilon(0) = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < \epsilon_1 \quad \text{and} \quad |z_2| < \epsilon_2\}.$$

(a) Describe the set $\partial B_\epsilon(0)$, the boundary of the polydisc $B_\epsilon(0)$, in a similar way as the above description of the polydisc itself.

(b) Let $\delta = (\delta_1, \delta_2)$ with $\delta_1, \delta_2 > 0$. Let $(a, b) \in \partial B_\epsilon(0)$. Show that the polydisc $B_\delta(a, b)$ around (a, b) contains both a point in $B_\epsilon(0)$ and a point in the complement of $B_\epsilon(0)$ by explicitly writing two such points.

Solution:

The boundary can be described as follows.

$$\partial B_\epsilon(0) = \{(z_1, z_2) \in \mathbb{C} \mid \text{either } |z_1| = \epsilon_1 \quad \text{and} \quad |z_2| \le \epsilon_2, \quad \text{or} \quad |z_1| \le \epsilon_1 \quad \text{and} \quad |z_2| = \epsilon_2\}$$

Next we show that this description satisfies the definition of boundary in the sense that any neighborhood of any point satisfying the above conditions contains both a point in $B_\epsilon(0)$ and a point in its complement.

Let $(a, b) \in \partial B_\epsilon(0)$ and assume that $|a| = \epsilon_1$. Define

$$(z_1, z_2) = ((1 + \frac{\delta_1}{2\epsilon_1})a, b).$$

Then $|z_1 - a| = (\frac{\delta_1}{2\epsilon_1})|a| = \frac{\delta_1}{2} < \delta_1$ and $|z_2 - b| = 0 < \delta_2$. Hence $(z_1, z_2) \in B_\delta(a, b)$. But $|z_1| = (1 + \frac{\delta_1}{2\epsilon_1})|a| = (1 + \frac{\delta_1}{2\epsilon_1})\epsilon_1 > \epsilon_1$, hence $(z_1, z_2) \notin B_\epsilon(0)$.

Next, choose a positive real number n such that both

$$0 \le 1 - \frac{\delta_i}{n\epsilon_i} < 1, \quad \text{for} \quad i = 1, 2.$$

Define

$$(z_1, z_2) = ((1 - \frac{\delta_1}{n\epsilon_1})a, (1 - \frac{\delta_2}{n\epsilon_2})b).$$

Since $|z_1 - a| < \delta_1$ and $|z_2 - b| < \delta_2$, we have that $(z_1, z_2) \in B_\delta(a, b)$. Moreover we have $|z_i| \le (1 - \frac{\delta_i}{n\epsilon_i})\epsilon_i < \epsilon_i$ for $i = 1, 2$. Hence $(z_1, z_2) \in B_\epsilon(0)$.

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Q-2) Let U be a non-empty open subset in $\mathbb{C}^n$ where $n > 1$, and let $f : U \to \mathbb{C}$ be a non-constant holomorphic function. Show that $f(U)$ is open in $\mathbb{C}$. You may use the fact that the statement is true when $n = 1$.

Solution:

Choose a point p in $F(U) \subset \mathbb{C}$. Let $q \in U$ be a preimage of p, i.e. $f(q) = p$. Choose an $\epsilon > 0$ such that if $\vec{\epsilon} = (\epsilon, \ldots, \epsilon)$ and $q = (q_1, \ldots, q_n)$, then the polydisc

$$B_{\vec{\epsilon}}(q) = \{z \in \mathbb{C}^n \mid |z_i - q_i| < \epsilon, \quad \text{for} \quad i = 1, \ldots, n\}$$

lies totally in U. Such a choice is possible since U is open. Let

$$D_\epsilon = \{w \in \mathbb{C} \mid |w| < \epsilon\}$$

be the open disc around the origin in $\mathbb{C}$ of radius ϵ. Define a collection of vectors $\lambda \in \mathbb{R}^n$ as

$$V = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid |\lambda_i| < 1 \quad \text{for} \quad i = 1, \ldots, n\}.$$

It follows that

$$z \in B_{\vec{\epsilon}}(q) \quad \text{if and only if} \quad z = q + \lambda w \quad \text{for some} \quad \lambda \in V \quad \text{and} \quad w \in D_\epsilon.$$

For each $\lambda \in V$ define the set

$$U_\lambda = \{z \in \mathbb{C}^n \mid z = q + \lambda w \quad \text{for some} \quad w \in D_\epsilon\}.$$

We claim that there exists a $\lambda \in V$ such that $f|U_\lambda$ is not constant. Assume the contrary. Since we have

$$B_{\vec{\epsilon}}(q) = \bigcup_{\lambda \in V} U_\lambda,$$

and since $q \in U_\lambda$ for every $\lambda \in V$, it then follows that $f|B_{\vec{\epsilon}}(q)$ is constant and is equal to $f(q)$. This contradicts the assumption that f is not constant.

Finally fix a $\lambda \in V$ such that f is not constant on U_λ.

Consider the function

$$\begin{aligned} \phi_\lambda : D_\epsilon &\to f(U) \subset \mathbb{C}, \\ w &\mapsto f(q + \lambda w). \end{aligned}$$

Note that $\phi_\lambda(D_\epsilon) = f(U_\lambda) \subset f(U)$. Since as a holomorphic function of a single variable, the function ϕ_λ is an open mapping, the set $\phi_\lambda(D_\epsilon)$ is an open neighborhood of p in $f(U)$. Hence f is an open mapping.

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Q-3) Let U be a non-empty open subset in $\mathbb{C}^n$ where $n > 1$, and let $f : U \to \mathbb{C}$ be a non-constant holomorphic function. Show that there is no point $z_0 \in U$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in U$. You may use the fact that the statement is true when $n = 1$.

Solution:

First Solution: Suppose there exists a $z_0 \in U$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in U$. Set $q = z_0$. Using the notation of the solution for Question 2, consider the one variable holomorphic function ϕ_λ for any fixed $\lambda \in V$. Then we would have $|\phi_\lambda(0)| \geq |\phi_\lambda(w)|$ for every $w \in D_\epsilon$, a clear contradiction.

Second Solution: Define $\phi : \mathbb{C} \to \mathbb{R}$ as $\phi(z) = |z|$. It is clear that ϕ is an open mapping. We showed in Question 2 that f is an open mapping. Composition of open mappings being open, the map $(\phi \circ f)$ is open. Hence $(\phi \circ f)(z) = |f(z)|$ cannot attain a maximum for $z \in U$.