Due Date: 12 May 2015, Tuesday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:..
STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
40	30	30	0	0	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you write your answers alone. Any similarity with your written words with any other solution or any other source that I happen to know is a direct violation of honesty.

(2) In particular do not lend your written solutions to your friends, nor borrow your friends's written solutions. Oral exchange of ideas is acceptable and is in fact encouraged.

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NAME: STUDENT NO:

Q-1) Prove the Cauchy Integral Formula for smooth functions on an annulus. That is, show that

$$f(z) = \frac{1}{2\pi i}\int_{\partial A_r} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{A_r} \frac{\partial f(w)}{\partial \bar{w}}\,\frac{dw\wedge d\bar{w}}{w-z}, \quad \text{for all} \quad z \in A_r,$$

where $0 < r < 1$, $A_r = \{z \in \mathbb{C} \mid r < |z| < 1\}$ and $f \in C^\infty(\bar{A}_r)$.
Also show how this formula reduces to the usual one when f is C^∞ in the closed unit disk.

Solution:

Consider the differential form

$$\eta = \frac{1}{2\pi i}\,\frac{f(w)}{w-z}\,dw.$$

For $w \neq z$, we have $\dfrac{\partial}{\partial \bar{w}}\left(\dfrac{1}{w-z}\right) = 0$, so

$$d\eta = -\frac{1}{2\pi i}\,\frac{\partial f(w)}{\partial \bar{w}}\,\frac{dw\wedge d\bar{w}}{w-z}.$$

Let $\Delta_\epsilon = \Delta(z,\epsilon)$ be an open disc of radius $\epsilon > 0$ around z totally lying in A_r. The form η is C^∞ in $A_r \setminus \Delta_\epsilon$.

By Stokes' theorem we have

$$\int_{A_r\setminus\Delta_\epsilon} d\eta = \int_{\partial(A_r\setminus\Delta_\epsilon)} \eta = \int_{\partial A_r} \eta - \int_{\partial \Delta_\epsilon} \eta.$$

Equivalently we get

$$\int_{\partial\Delta_\epsilon} \eta = \int_{\partial A_r} \eta - \int_{A_r\setminus\Delta_\epsilon} d\eta \tag{A}$$

or

$$\int_{\partial\Delta_\epsilon} \eta = \int_{\partial A_r} \eta - \int_{A_r} d\eta + \int_{\Delta_\epsilon} d\eta. \tag{B}$$

We now investigate the integrals which involve ϵ separately.

(1) Set $w - z = re^{i\theta}$. Here r is the variable in polar coordinates, not the fixed radius of the annulus A_r. Using this we get

$$\int_{\partial\Delta_\epsilon} \eta = \frac{1}{2\pi i}\int_{\partial\Delta_\epsilon} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})\,d\theta,$$

which goes to $f(z)$ when $\epsilon \to 0$.

(2) Since $\dfrac{\partial f(w)}{\partial \bar{w}}$ is continuous on $\bar{A}_r$, its absolute value is bounded there, say by a real number $c > 0$. After passing to polar coordinates as above we then have

$$|d\eta| = \frac{1}{2\pi}\left|\frac{\partial f(w)}{\partial \bar{w}}\,\frac{dw \wedge d\bar{w}}{w - z}\right| = \frac{1}{\pi}\left|\frac{\partial f(w)}{\partial \bar{w}}\,dr \wedge d\theta\right| \leq \frac{c}{\pi}|dr \wedge d\theta|,$$

which gives

$$\left|\int_{\Delta_\epsilon} d\eta\right| \leq \frac{c}{\pi}\int_{\Delta_\epsilon} |dr \wedge d\theta| = 2c\epsilon,$$

which in turn goes to zero as ϵ goes to zero.

Now from equation (B), by taking limit as ϵ goes to zero of both sides, we get

$$f(z) = \int_{\partial A_r} \eta - \int_{A_r} d\eta,$$

which is precisely the formula we set out to prove.

Now assume that f is C^∞ on the closure of the unit disc Δ. Let Δ_r denote the open disc around the origin with radius r. We then have the obvious relations

$$\partial A_r = \partial\Delta \setminus \partial\Delta_r,\ A_r \setminus \Delta_\epsilon = (\Delta \setminus \Delta_\epsilon) \setminus \bar{\Delta}_r, \quad \text{and} \quad \partial\bar{\Delta}_r = \partial\Delta_r,$$

where $\bar{\Delta}_r$ denotes the closure.

Now starting with equation (A) we have

$$\begin{aligned}
\int_{\partial\Delta_\epsilon} \eta &= \int_{\partial A_r} \eta - \int_{A_r \setminus \Delta_\epsilon} d\eta \\
&= \int_{\partial\Delta} \eta - \int_{\partial\Delta_r} \eta - \int_{\Delta\setminus\Delta_\epsilon} d\eta + \int_{\bar{\Delta}_r} d\eta \\
&= \int_{\partial\Delta} \eta - \int_{\partial\Delta_r} \eta - \int_{\Delta\setminus\Delta_\epsilon} d\eta + \int_{\partial\Delta_r} \eta \\
&= \int_{\partial\Delta} \eta - \int_{\Delta\setminus\Delta_\epsilon} d\eta,
\end{aligned}$$

where in the third line we used Stokes' theorem

$$\int_{\bar{\Delta}_r} d\eta = \int_{\partial\Delta_r} d\eta.$$

Now the above arguments, applied verbatim, give

$$f(z) = \int_{\partial\Delta} \eta - \int_{\Delta} d\eta,$$

which is the usual Cauchy Integral Formula (CIF) for smooth f on the closed unit disk.

Remark: Notice that in the above discussion, the outer radius of A_r was not involved in any way during the proof. This prompts the following corollary which we will use in the next question.

Corollary: For any $0 < r < R$, define the annulus

$$B_{r,R} = \{z \in \mathbb{C} \mid r < |z| < R\}.$$

Then for any $f \in C^\infty(\bar{B}_{r,R})$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_{r,R}} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{B_{r,R}} \frac{\partial f(w)}{\partial \bar{w}}\frac{dw \wedge d\bar{w}}{w-z}, \quad \text{for all} \quad z \in B_{r,R}.$$

NAME: STUDENT NO:

Q-2) Prove the $\bar{\partial}$-Poincare lemma for the punctured disk. That is, for any $g \in C^\infty(\Delta^*)$, there exists an $f \in C^\infty(\Delta^*)$ such that

$$\frac{\partial f(z)}{\partial \bar{z}} = g(z), \quad \text{for} \quad z \in \Delta^*,$$

where

$$\Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}.$$

Solution:

We first prove the $\bar{\partial}$-Poincare Lemma for the annulus $B_{r,R}$: For every $g(z) \in C^\infty(\bar{B}_{r,R})$, the function

$$f(z) = \frac{1}{2\pi i} \int_{B_{r,R}} \frac{g(w)}{w - z} \, dw \wedge d\bar{w}$$

is defined and C^∞ in $B_{r,R}$ and satisfies

$$\frac{\partial f}{\partial \bar{z}} = g.$$

The proof of this claim follows almost verbatim the proof given for Δ in Griffiths and Harris on page 5.

For every $z_0 \in B_{r,R}$ choose $\epsilon > 0$ such that the disc $\Delta(z_0, 2\epsilon)$ lies in $B_{r,R}$. Consider a covering of $B_{r,R}$ by the open sets $\Delta(z_0, \epsilon)$. We will show that the above claim holds on every $\Delta(z_0, \epsilon)$.

Fix one $\Delta(z_0, \epsilon)$. Let $U_1 = \Delta(z_0, 2\epsilon)$ and $U_2 = A_r \setminus \Delta(z_0, \epsilon)$. Let $\rho_1(z) + \rho_2(z) = 1$ be a partition of unity subordinate to the open covering $A_r = U_1 \cup U_2$ with the support of ρ_i being in U_i. Define $g_i = \rho_i g$. Then we have

$$g(z) = g_1(z) + g_2(z),$$

with $g_1(z)$ vanishing outside $\Delta(z_0, 2\epsilon)$ and $g_2(z)$ vanishing inside $\Delta(z_0, \epsilon)$.

For every $w \in B_{r,R}$, the function $\dfrac{g_2(w)}{w - z}$ is a holomorphic function of z for $z \in \Delta(z_0, \epsilon)$. In particular $\dfrac{\partial}{\partial \bar{z}} \dfrac{g_2(w)}{w - z} = 0$ on $\Delta(z_0, \epsilon)$. If we now define

$$f_2(z) = \frac{1}{2\pi i} \int_{B_{r,R}} g_2(w) \frac{dw \wedge d\bar{w}}{w - z}, \quad \text{for} \quad z \in \Delta(z_0, \epsilon),$$

then the above arguments show that f_2 is well defined and in fact holomorphic. Hence $\dfrac{\partial f_2(z)}{\partial \bar{z}} = 0.$

Now define

$$f_1(z) = \frac{1}{2\pi i} \int_{B_{r,R}} g_1(w) \frac{dw \wedge d\bar{w}}{w - z}, \quad \text{for} \quad z \in \Delta(z_0, \epsilon).$$

We will show that $f_1(z)$ is well defined and C^∞ in $B_{r,R}$, and that moreover satisfies $\dfrac{\partial f_1(z)}{\partial \bar{z}} = g_1(z)$ for $z \in \Delta(z_0, \epsilon)$.

For this note that g_1 vanishes outside $\Delta(z_0, 2\epsilon)$, so we can replace $B_{r,R}$ in the integral with $\mathbb{C}$ without changing the value of $f_1(z)$. Next substitute $u = w - z$ in the integral to obtain

$$f_1(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} g_1(u+z) \frac{du \wedge d\bar{u}}{u}.$$

Putting $u = x + iy$, we get

$$du \wedge d\bar{u} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy = -2irdr \wedge d\theta,$$

where in the last equation we passed to polar coordinates. Note that this r is the parameter in polar coordinates and is not the inner radius of $B_{r,R}$. We can now write

$$f_1(z) = -\frac{1}{\pi} \int_{\mathbb{C}} g_1(z + re^{i\theta}) e^{-i\theta} dr \wedge d\theta,$$

which is now clearly defined and C^∞ on $\Delta(z_0, \epsilon)$. Moreover we have

$$\frac{\partial f_1(z)}{\partial \bar{z}} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial g_1(z + re^{i\theta})}{\partial \bar{z}} e^{-i\theta} dr \wedge d\theta.$$

Putting back all the substitutions we made, we get

$$\frac{\partial f_1(z)}{\partial \bar{z}} = \frac{1}{2\pi i} \int_{B_{r,R}} \frac{g_1(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}.$$

On the other hand, since $g_1(w)$ vanishes outside of $\Delta(z_0, \epsilon)$, we clearly have

$$\frac{1}{2\pi i} \int_{\partial B_{r,R}} \frac{g_1(w)}{w - z} dw = 0.$$

We then have

$$\frac{\partial f_1(z)}{\partial \bar{z}} = \frac{1}{2\pi i} \int_{\partial B_{r,R}} \frac{g_1(w)}{w - z} dw + \frac{1}{2\pi i} \int_{B_{r,R}} \frac{g_1(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z} = g_1(z),$$

where in the last equation we used the generalized version of CIF for $B_{r,R}$, which we proved as a corollary at the end of question 1.

Finally, we have

$$\begin{aligned} \frac{f(z)}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(f_1(z) + f_2(z)) & \\ &= g_1(z) + 0 \\ &= g(z) \quad \text{for} \quad z \in \Delta(z_0, \epsilon). \end{aligned}$$

This then completes the proof of the $\bar{\partial}$-Poincare Lemma for $B_{r,R}$.

Now we prove the $\bar{\partial}$-Poincare Lemma for the punctured disc.

We will use special values for r and R. We define for every integer $k \geq 3$

$$B_k = B_{\frac{1}{k},1-\frac{1}{k}} = \{z \in \mathbb{C} \mid \frac{1}{k} < |z| < 1 - \frac{1}{k}\}.$$

Since $g \in C^\infty(\Delta^*)$, we must have $g \in C^\infty(\bar{B}_k)$ for every $k \geq 3$, so by the above argument we know that there exists $f_k \in C^\infty(B_k)$ such that $\dfrac{\partial f_k}{\partial \bar{z}} = g$ on B_k.

We first prove that for every $k > 3$, there exists $\alpha_k \in C^\infty(\Delta^*)$ such that $\dfrac{\partial \alpha_k}{\partial \bar{z}} = g$ on B_k. Fix $k > 3$. Let $\alpha \in C^\infty(B_{k+1})$ such that $\dfrac{\partial \alpha}{\partial \bar{z}} = g$ on B_{k+1}. Choose a C^∞ bump function ρ which is $\equiv 1$ on B_k, and has compact support in B_{k+1}. Set $\alpha_k = \rho\alpha$. Now we have $\alpha_k \in C^\infty(\Delta^*)$ and $\dfrac{\partial \alpha_k}{\partial \bar{z}} = g$ on B_k. Here α_k is actually C^∞ on all of the unit disk but we will be restricting it later to the punctured disk so we might as well start here.

Now we will construct a sequence of functions $f_4, f_5, \ldots$, all in $C^\infty(\Delta^*)$ such that (i) for each k, we have $\dfrac{\partial f_k}{\partial \bar{z}} = g$ on B_k, and (ii) the sequence converges uniformly on compact subsets of Δ^* to a function $f \in C^\infty(\Delta^*)$. It then follows that $\dfrac{\partial f}{\partial \bar{z}} = g$ on Δ^*.

Fix an arbitrary $\epsilon > 0$.

We choose $f_4 \in C^\infty(\Delta^*)$ as above so that $\dfrac{\partial f_4}{\partial \bar{z}} = g$ on B_4. To construct f_5, first choose $\alpha \in C^\infty(\Delta^*)$ such that $\dfrac{\partial \alpha}{\partial \bar{z}} = g$ on B_5. The difference $f_4 - \alpha$ is now holomorphic on B_4, so has a Laurent expansion around the origin converging in B_4. Truncate this Laurent expansion from below and above to obtain a rational function β which is necessarily holomorphic on Δ^* such that

$$\sup_{B_3} |(f_4 - \alpha) - \beta| < \frac{\epsilon}{2^4}.$$

Set $f_5 = \alpha + \beta$. Clearly $f_5 \in C^\infty(\Delta^*)$ and $\dfrac{\partial f_5}{\partial \bar{z}} = \dfrac{\partial \alpha}{\partial \bar{z}} = g$ in B_5.

Repeating the above argument we construct the sequence $\{f_k\}$ such that for $k > 3$,

$$f_k \in C^\infty(\Delta^*) \quad \text{with} \quad \frac{\partial f_k}{\partial \bar{z}} = g \quad \text{on} \quad B_k \quad \text{and} \quad \sup_{B_{k-1}} |f_{k+1} - f_k| < \frac{\epsilon}{2^k}.$$

To show the uniform convergence on compact subsets of Δ^* let D be such a compact set. Let k be such that $D \subset B_k$. For any integers $n > m > k$ we have

$$
\begin{aligned}
\sup_D |f_n - f_m| &\leq \sup_{B_k} |f_n - f_m| \\
&= \sup_{B_k} |(f_n - f_{n-1}) + (f_{n-1} - f_{n-2}) + \cdots + (f_{m-1} - f_m)| \\
&\leq \sum_{\ell=m}^{n-1} \sup_{B_k} |f_{\ell+1} - f_\ell| \\
&\leq \sum_{\ell=m}^{n-1} \sup_{B_{\ell-1}} |f_{\ell+1} - f_\ell| \\
&< \sum_{\ell=m}^{n-1} \frac{\epsilon}{2^\ell} \\
&< \epsilon.
\end{aligned}
$$

Thus the sequence $\{f_k\}$ converges uniformly on compacta to a function $f \in C^\infty(\Delta^*)$ such that

$$\frac{\partial f}{\partial \bar{z}} = g \quad \text{on} \quad \Delta^*.$$

This last statement follows from functional analysis, see for example Rudin's *Functional Analysis*.

Corollary: $H^{0,1}(\Delta^*) = 0$.
Proof: Let α be a $\bar{\partial}$-closed $(0,1)$-form on Δ^*. Then α is of the form $\alpha = \alpha_0 \, d\bar{z}$ for some $\alpha_0 \in C^\infty(\Delta^*)$. By the above solution we know that there exists a $(0,0)$-form $f \in C^\infty(\Delta^*)$ such that $\dfrac{\partial f}{\partial \bar{z}} = \alpha_0$ on Δ^*. Hence every $\bar{\partial}$-closed $(0,1)$-form is exact. Hence the cohomology is zero.

NAME: STUDENT NO:

Q-3) Show that every holomorphic line bundle on the unit punctured disc in the plane is trivial.

Solution:

Since $\mathrm{Pic}(\Delta^*) = H^1(\Delta^*, \mathcal{O}^*)$, we need to show that this cohomology is zero.

Consider the exact sequence of sheaves on Δ^*

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

which gives rise to the exact cohomology sequence

$$\cdots \to H^1(\Delta^*, \mathcal{O}) \to H^1(\Delta^*, \mathcal{O}^*) \to H^2(\Delta^*, \mathbb{Z}) \to \cdots .$$

Since Δ^* is homotopic to S^1, the real unit circle in the plane and since $\dim_{\mathbb{R}} S^1 = 1$, its higher cohomology groups vanish, i.e. we have

$$H^2(\Delta^*, \mathbb{Z}) = H^2(S^1, \mathbb{Z}) = 0.$$

Moreover we have

$$H^1(\Delta^*, \mathcal{O}) \cong H^{0,1}(\Delta^*) = 0$$

as we showed in the previous question.

It then follows from the above exact sequence of cohomology that $H^1(\Delta^*, \mathcal{O}^*) = 0$ as desired.