Due Date: 3 March 2017, Thursday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Homework 2 – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me, or even with people who took this course before. It is not considered good behavior to ask these questions at online forums without mentioning that these are homework questions of an introductory course.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) **It is absolutely mandatory that you write your answers alone.**

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. *(It is always nice to flatter your friends by using their ideas and thanking them.)*

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

(6) Do not lend your written work to your friends and do not ask to borrow their written work. You may explain your solutions to your friends to any degree of detail you like, or you may ask them as many questions as they are willing to answer. **But the final writing process should be done alone.**

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work. Moreover I declare that every solution I wrote reflects my true understanding of the problem, and any sources used, including ideas from friends, are explicitly cited without exception.

Please sign here: *Date:*

NAME: STUDENT NO:

Q-1) Let $U \subset \mathbb{C}^n$, $n \geq 1$, be a non-empty, open and connected subset which contains a nowhere dense closed subset Z such that $U \setminus Z$ is disconnected, i.e. there are open non-empty subsets $A, B \subset U$ such that $U \setminus Z = A \cup B$ with $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Show that $\bar{A} \cap \bar{B} \cap Z$ cannot be empty.

Give an example of U and Z as above such that $U \setminus Z$ is disconnected.

Now let Z be the zero set of an analytic function. Show that $U \setminus Z$ is always connected.

Solution:

First observe that if $p \in \bar{A} \cap \bar{B}$, then p belongs to Z. Assume $p \in U$. Then p is either in A or B. Without loss of generality say $p \in A$. But p is also in $\bar{B}$ so $p \in A \cap \bar{B}$ which is a contradiction since $A \cap \bar{B}$ is empty. So $\bar{A} \cap \bar{B}$ is a subset of Z. If $\bar{A} \cap \bar{B} \cap Z$ is empty then $\bar{A} \cap \bar{B}$ must be empty.

Since we have $U \setminus Z = A \cup B$, taking the closure of both sides within the topology of U, we get $U = \bar{A} \cup \bar{B}$, since the interior of Z is empty. Since $\bar{A} \cap \bar{B}$ is empty, this shows that U must be disconnected contrary to our assumption. So $\bar{A} \cap \bar{B} \cap Z$ cannot be empty.

As an example let U be the open ball around the origin in the Euclidean topology. If $z_j = x_j + iy_j$ then

$$U = \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{C}^n = \mathbb{R}^{2n} \mid x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2 < 1\}.$$

Now define

$$Z = \{(x_1, y_1, \ldots, x_n, y_n) \in U \mid y_n = 0\}.$$

Then $U \setminus Z$ is disconnected; Setting

$$A = \{(x_1, y_1, \ldots, x_n, y_n) \in U \mid y_n > 0\} \quad \text{and} \quad B = \{(x_1, y_1, \ldots, x_n, y_n) \in U \mid y_n < 0\},$$

we see that $U \setminus Z = A \cup B$, and $\bar{A} \cap \bar{B} \cap Z \supset Z$, so is not empty.

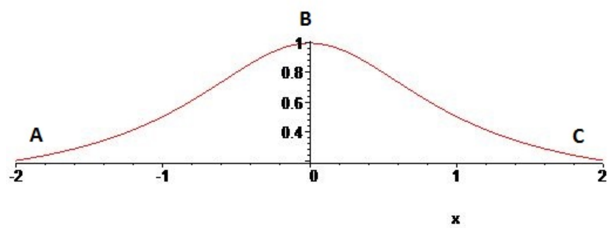
If Z is the zero set of an analytic function, then it is clearly closed and nowhere dense. Suppose $U \setminus Z = A \cup B$ as above. Define an analytic function f on $U \setminus Z$ as follows: $f(p) = 0$ if $p \in A$, and $f(p) = 1$ if $p \in B$. By the Riemann extension theorem since f is bounded around Z, it can be analytically extended over z. But there is a point $p \in \bar{A} \cap \bar{B} \cap Z$, and clearly f cannot be continuously defined at p. This contradiction shows that $U \setminus Z$ is always connected when Z is the zero set of an analytic function.

NAME: STUDENT NO:

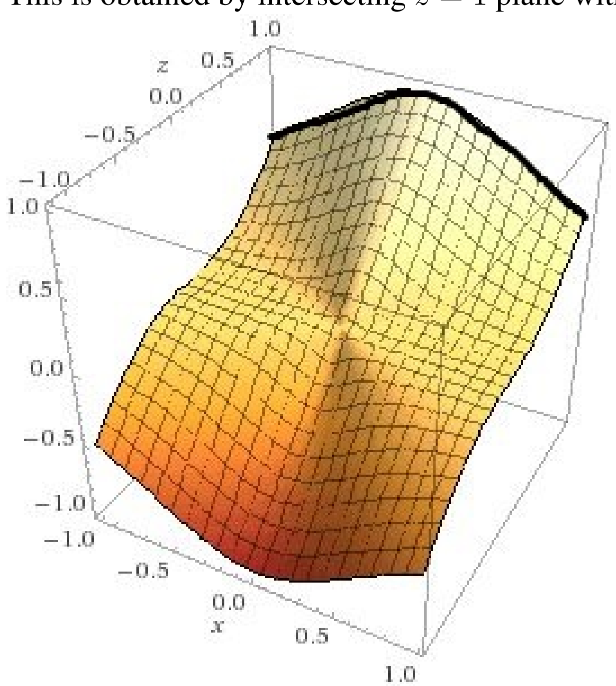
Q-2) Let X be the real projective variety defined in $\mathbb{P}^2_{\mathbb{R}}$ by the homogeneous equation $yx^2 + yz^2 = z^3$. Using a computer program, such as WolframAlpha, plot the graph of X in each fundamental affine coordinate chart.

Solution:

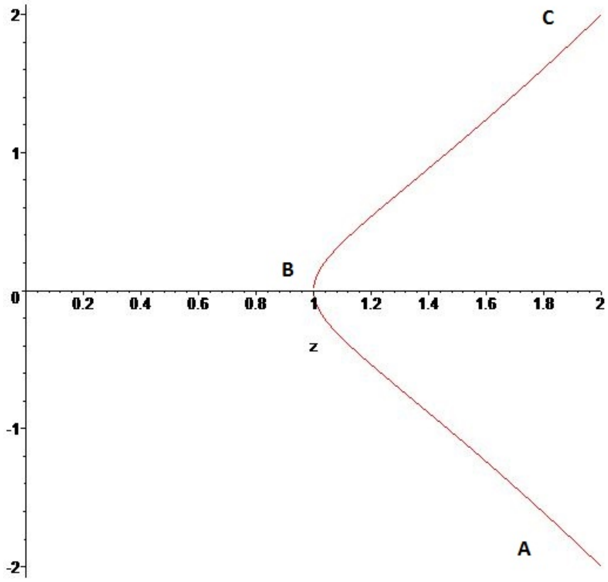
First consider the chart $z = 1$. In this chart our curve looks like as follows. Here the arms A and C go towards infinity which we will see in other charts.



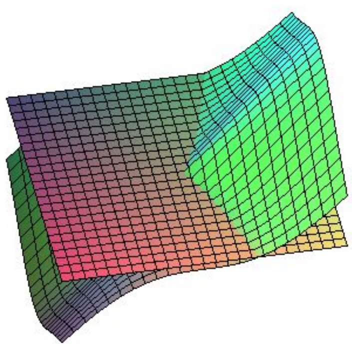
This is obtained by intersecting $z = 1$ plane with the $yx^2 + yz^2 - z^3 = 0$ surface (cone) in $\mathbb{R}^3$.



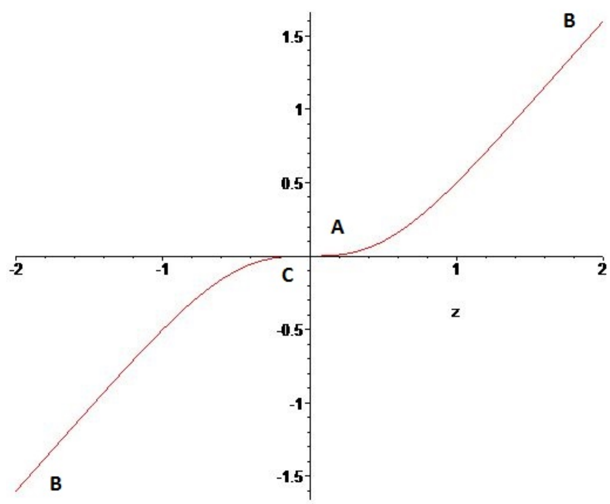
In the chart $y = 1$ our curve looks like as follows. Note that the point B, the top of the previous curve can be seen in this chart but the arms A and C still go to infinity which is not seen in this chart.



This is obtained by intersecting $x = 1$ plane with the $yx^2 + yz^2 - z^3 = 0$ surface (cone) in $\mathbb{R}^3$.



In the chart $x = 1$ our curve looks like as follows. Here the arms A and C finally reach infinity which is the origin in this chart. But the tip B of the curve of chart $z = 1$ is now infinity for us so we anly see the arms going towards that infinity.



This is obtained by intersecting $x = 1$ plane with the $yx^2 + yz^2 - z^3 = 0$ surface (cone) in $\mathbb{R}^3$.