Due Date: 17 March 2017, Friday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:...

STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Midterm 1 – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me, or even with people who took this course before. It is not considered good behavior to ask these questions at online forums without mentioning that these are homework questions of an introductory course.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) **It is absolutely mandatory that you write your answers alone.**

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. *(It is always nice to flatter your friends by using their ideas and thanking them.)*

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

(6) Do not lend your written work to your friends and do not ask to borrow their written work. You may explain your solutions to your friends to any degree of detail you like, or you may ask them as many questions as they are willing to answer. **But the final writing process should be done alone.**

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work. Moreover I declare that every solution I wrote reflects my true understanding of the problem, and any sources used, including ideas from friends, are explicitly cited without exception.

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Q-1) State the "Analytic Implicit Function Theorem" for holomorphic maps, and cite your source explicitly. Next let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and let $Z(f)$ be its zero set in $\mathbb{C}^n$. State the conditions for $Z(f)$ to be a complex manifold and prove your claim.

Solution:

The statement can be looked up from any introductory text. For the second part, as an application of both the implicit function theorem and the inverse function theorem, the zero set of f will be a manifold if and only if at each point of the zero set at least one partial derivative of f is non-zero.

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Q-2) Let ℓ be a line in $\mathbb{C}^{n+1}$ through the origin. Note that $[\ell]$ is a point of $\mathbb{P}^n$. Let S^{2n+1} be the unit $(2n+1)$-sphere centered at the origin in $\mathbb{R}^{2n+2}$. Let $P(\ell)$ denote the plane in $\mathbb{R}^{2n+2}$ through the origin corresponding to the complex line ℓ. Show that $P(\ell)$ intersects S^{2n+1} in a circle. Moreover show that if ℓ_1 and ℓ_2 are two distinct lines in $\mathbb{C}^{n+1}$ through the origin, then $P(\ell_1) \cap S^{2n+1}$ and $P(\ell_2) \cap S^{2n+1}$ are two disjoint circles.
Now solve Exercise 2.1.1 on Huybrechts's book: Show that $\mathbb{P}^n$ is a compact complex manifold. Describe a diffeomorphism of $\mathbb{P}^1$ with the two dimensional sphere S^2. Conclude that $\mathbb{P}^1$ is simply connected.

Solution:

$P(\ell)$ is a real 2-plane and when it intersects S^{2n+1} the intersection points on $P(\ell)$ all have length 1, so lie on the unit circle centered at the origin on $P(\ell)$.

Let $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ be a non-zero point. It defines a line ℓ_z in $\mathbb{C}^{n+1}$ passing through the origin. The points of L_z are of the form λz for $\lambda \in \mathbb{C}$. Let $z_j = x_j + iy_j$ where x_j and y_j are real. Notice that the following points lie in the real plane $P(\ell_z)$ in $\mathbb{R}^{2n+2}$.

$$z = (x_0, y_0, x_1, y_1, \ldots, x_{n+1}, y_{n+1}),$$
$$iz = (-y_0, x_0, -y_1, x_1, \ldots, -y_{n+1}, x_{n+1}).$$

Thus every point of $\mathbb{P}^n$, being a complex line through the origin in $\mathbb{C}^{n+1}$ defines a 2-plane in $\mathbb{R}^{2n+2}$ through the origin but not every plane through the origin in $\mathbb{R}^{2n+2}$ is determined by a point of $\mathbb{P}^n$. For example consider the plane P spanned over $\mathbb{R}$ by the standard basis vectors

$$e_1 = (1, 0, \ldots, 0) \quad \text{and} \quad e_3 = (0, 0, 1, 0, \ldots, 0).$$

If this plane was defined by a point of $\mathbb{P}^n$,then the point

$$ie_1 = (0, 1, 0, \ldots, 0)$$

would be in P. But ie_1 does not belong to P, so P is not determined by a point of $\mathbb{P}^n$.

Now we define an action of $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ on S^{2n+1} as follows. For $z = (z_0, \ldots, z_{n+1}) \in S^{2n+1}$ as above and $\lambda \in S^1$, we define

$$\lambda \cdot z = (\lambda z_0, \ldots, \lambda z_{n+1}).$$

Let z be an arbitrary point on S^{2n+1}, and let $S^1 z$ be the orbit of z under the action of S^1. Note that $iz \in S^2 z$. Observe that by passing to real coordinated we have, when $\lambda = \lambda_1 + i\lambda_2 \in S^1$,

$$z = u = (x_0, y_0, \ldots, x_{n+1}, y_{n+1}), \tag{1}$$
$$iz = v = (-y_0, x_0, \ldots, -y_{n+1}, x_{n+1}), \tag{2}$$
$$\lambda \cdot z = \lambda_1 \, u + \lambda_2 \, v \in \operatorname{span}_{\mathbb{R}}(u, v). \tag{3}$$

Thus to every $z \in S^{2n+1}$ corresponds a unique plane in $\mathbb{R}^{2n+2}$ passing through the origin and defining a point in $\mathbb{P}^n$.

Moreover for two points $z_1, z_2 \in S^{2n+1}$, if there exists a point $w \in S^1 z_1 \cap S^1 z_2$, then $w = \alpha z_1 = \beta z_2$ for some $\alpha, \beta \in S^1$. Hence $z_1 = \alpha^{-1} \beta z_2$ and consequently $S^1 z_1 = S^1 z_2$.

Now back to business! We showed that there is a one-to-one onto map

$$S^{2n+1}/S^1 \longrightarrow \mathbb{P}^n.$$

This map is continuous once we give both sides the corresponding quotient topologies. Finally, since S^{2n+1} is compact, so is $\mathbb{P}^n$.

As for a diffeomorphism of S^2 with P^2, the basic idea is to pass from S^2 to $\mathbb{C}$ with the stereographic projection and embed $\mathbb{C}$ into $\mathbb{P}^1$. Here are the details.

Identify S^2 with the unit sphere $x^2 + y^2 + z^2 = 1$. Every point on S^2 can thus be written as

$$(\sqrt{1 - z^2}\cos\theta, \sqrt{1 - z^2}\sin\theta, z), \quad \text{for some } \theta.$$

Under the stereographic projection this point goes to

$$\sqrt{\frac{1+z}{1-z}}e^{i\theta} \in \mathbb{C}.$$

Sending it to $\mathbb{P}^2$ we get the point

$$[\sqrt{1+z}e^{i\theta} : \sqrt{1-z}] \in \mathbb{P}^2.$$

Conversely, given any

$$[Re^{i\theta} : 1] \in \mathbb{P}^2,$$

we set

$$z = \frac{R^2 - 1}{R^2 + 1},$$

and this point is in the image of

$$(\sqrt{1 - z^2}\cos\theta, \sqrt{1 - z^2}\sin\theta, z) \in S^2.$$

This establishes the required diffeomorphism.

It follows that since S^2 is simply connected, so is $\mathbb{P}^1$.