Bilkent University

Exam # 02
Math 430 Introduction to Complex Geometry
Due: 20 April 2020
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Solution Key

Q-1) In this question we are using the notation of the Hard Lefschetz Theorem, [Griffiths-Harris, PAG, p122].
Here is a reminder about the notation.
M is a compact, complex, Hermitian manifold of complex dimension n.
$L : A^p(M) \to A^{p+2}(M)$, where $L(\alpha) = \alpha \wedge \omega$, with $\alpha \in A^p(M)$ and ω is the associated $(1,1)$-form of the metric of M.
$\Lambda : A^p(M) \to A^{p-2}(M)$ is the adjoint of L.
$h : A^*(M) \to A^*(M)$, where if $\alpha = \sum_{p=0}^{2n} \alpha_p$, with $\alpha_p \in A^p(M)$, then $h(\alpha) = \sum_{p=0}^{2n}(n-p)\alpha_p$.
Recal that we have the relations,

$$[\Lambda, L] = h, \quad [h, L] = -2L, \quad [h, \Lambda] = 2\Lambda.$$

(a) Show that for any positive integer m, we have

$$[\Lambda, L^m] = mhL^{m-1} + m(m-1)L^{m-1},$$

where L^0 is defined as the identity map. In fact you can simplify this expression as

$$[\Lambda, L^m](\alpha) = m(n-k-m+1)L^{m-1}(\alpha),$$

where $\alpha \in A^k(M)$.

(b) Show that for any $\alpha \in H^{n-k}(M)$, we have $L^{k+1}(\alpha) = 0$ if and only if $\Lambda(\alpha) = 0$.

Solution
(a) We do this by induction. The $m = 1$ case is already given. Assume for m and check for $m + 1$. We will use the above given identities as $\Lambda L = h + L\Lambda$ and $Lh = hL + 2L$.

$$\begin{aligned}
[\Lambda, L^{m+1}] &= \Lambda L^{m+1} - L^{m+1}\Lambda \\
&= (\Lambda L)L^m - L^{m+1}\Lambda \\
&= (h + L\Lambda)L^m - L^{m+1}\Lambda \\
&= hL^m + L(\Lambda L^m - L^m\Lambda) \\
&= hL^m + L(mhL^{m-1} + m(m-1)L^{m-1}) \\
&= hL^m + m(Lh)L^{m-1} + m(m-1)L^m \\
&= hL^m + m(hL + 2L)L^{m-1} + m(m-1)L^m \\
&= (m+1)hL^m + (m+1)mL^m,
\end{aligned}$$

as required.
Moreover, if $\alpha \in H^{n-k}(M)$, then $L^{m-1}\alpha \in H^{k+2m-2}(M)$.

Then $hL^{m-1}\alpha = (n-k-2m+2)L^{m-1}\alpha$, and hence

$$\begin{aligned}[\Lambda, L^m]\alpha &= mhL^{m-1}\alpha + m(m-1)L^{m-1}\alpha \\ &= m(n-k-2m+2)L^{m-1}\alpha + m(m-1)L^{m-1}\alpha \\ &= m(n-k-m+1)L^{m-1}\alpha,\end{aligned}$$

where $\alpha \in A^k(M)$.

(b) This can be proved in several ways since there are numerous Kahler identities to use. Here I give a simple proof using the facts that

$$L^{k+2} : H^{n-k-2}(M) \to H^{n+k+2}(M),$$

and its adjoint

$$\Lambda^{k+2} : H^{n+k+2}(M) \to H^{n-k-2}(M)$$

are isomorphisms by the Hard Lefschetz Theorem.

Now let $\alpha \in H^{n-k}(M)$. By part (a) we have

$$[\Lambda, L^{k+1}]\alpha = (k+1)\overbrace{(n-(n-k)-(k+1)+1)}^{0}L^k\alpha = 0.$$

Thus we have

$$\Lambda L^{k+1}\alpha = L^{k+1}\Lambda\alpha.$$

First assume that $L^{k+1}\alpha = 0$. This gives $L^{k+1}\Lambda\alpha = 0$. Acting with L on this gives $L^{k+2}\Lambda\alpha = 0$, and since here L^{k+2} is an isomorphism, we get $\Lambda\alpha = 0$.

Conversely if $\Lambda\alpha = 0$, then we get $\Lambda L^{k+1} = 0$. Acting on this with Λ^{k+1} gives $\Lambda^{k+2}L^{k+1}\alpha = 0$. Now again, since Λ^{k+2} is an isomorphism, we get $L^{k+1}\alpha = 0$.