Bilkent University

Exam # 04
Math 430 Introduction to Complex Geometry
Due: 12 June 2020
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Q-1) Let $M \subset \mathbb{P}^n$ be an irreducible, m-dimensional smooth submanifold. Assume that M is nondegenerate, i.e. it does not lie in any hyperplane of $\mathbb{P}^n$. Let the degree of M be d. We assume that $n, d, m \geq 1$, and of course $n > m$. Show that $d + m \geq n + 1$.

Solution-1 There are several equivalent definitions of degree but we want to use the following one here. We intersect M with a generic $\mathbb{P}^{n-m}$ in $\mathbb{P}^n$. Then the number of intersection points is d, and is called the degree of M. here we assumed as in the question that the dimension of M is m.

We will prove the claim by induction on the dimension m of M.

First let $m = 1$. Any n points on M lie in a hyperplane H of $\mathbb{P}^n$. (To see this let $p_0, \ldots, p_{n-1}$ be n points on M. The lines $p_0p_1, \ldots, p_0p_{n-1}$ generate a linear subspace of dimension at most $n-1$, and hence lie in a hyperplane.) If the degree d of M is strictly less than n, then $H \cap M$, containing more than d points, must contain a components of M. But M is irreducible, so it must totally lie in H, contradicting our assumption that it is nondegenaerate. Hence the claim holds when $m = 1$.

Now let $m \geq 2$. Before we start the induction step, recall that for a generic hyperplane H in $\mathbb{P}^n$, the intersection $H \cap M$ is also smooth by Bertini's theorem and clearly $\deg M = \deg H \cap M$.

Moreover for a generic hyperplane H, the intersection $H \cap M$ is also nondegenerate. To show this choose n points $p_1, \ldots, p_n$ on M and let H be a hyperplane passing through these points. The points $p_1, \ldots, p_n$ also lie in $H \cap M$. If $H \cap M$ is degenerate, then there exists another hyperplane H' passing through these n points and containing $H \cap M$. But this means that the matrix whose rows are homogeneous coordinates of the points $p_1, \ldots, p_n$ does not have maximal rank. This is a closed condition on the space $\mathbb{P}^{n*}$ of hyperplanes of $\mathbb{P}^n$. Hence there is a Zariski open subset U of $\mathbb{P}^{n*}$ such that for any $H \in U$, the intersection $H \cap M$ is nondegenerate.

It is also easy to see that for a generic hyperplane H, the intersection $H \cap M$ is irreducible. To see this we notice by the Lefschetz hyperplane section theorem that $H_0(H \cap M, \mathbb{C}) \cong H_0(M, \mathbb{C}) \cong \mathbb{C}$. This shows that $H \cap M$ is connected. If it had more than one irreducible component, then the points where these components intersect one another would be singular points of M, contradicting the fact that $H \cap M$ is smooth for a generic H.

Finally, for the induction step we assume that the claim is true for all dimensions less than m. But the above observation says that when M is smooth, irreducible and nondegenerate of dimension m, then for a generic hyperplane H the intersection $H \cap M$ is smooth, irreducible and nondegenerate of dimension $m-1$. But $H \cap M$ lies in $H \cong \mathbb{P}^{n-1}$. By the induction assumption we have

$$d \geq (n-1) - (m-1) + 1 = n - m + 1,$$

as claimed. (See Griffiths and Harris, pages 173+.)

Q-2) A variety M in $\mathbb{P}^n$ is called normal if the linear system of M giving the embedding $M \hookrightarrow \mathbb{P}^n$ is complete. Prove that any hypersurface M in $\mathbb{P}^n$ of degree $d > 1$ is normal. (Here $n > 1$.)

Solution-2 We need to prove that for a hyperplane H of $\mathbb{P}^n$, the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}(H)) \xrightarrow{r} H^0(M, \mathcal{O}_M(H))$$

is surjective.

Start with the short exact sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(H - M) \to \mathcal{O}_{\mathbb{P}^n}(H) \to \mathcal{O}_M(H) \to 0,$$

where $\mathcal{O}_{\mathbb{P}^n}(H - M)$ is the sheaf of sections of H vanishing on M. The associated long exact cohomology sequence begins as

$$0 \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H-V)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H)) \xrightarrow{r} H^0(M, \mathcal{O}_M(H)) \to H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H-M)) \to \dots$$

We have

$$\mathcal{O}_{\mathbb{P}^n}(H - M) = \mathcal{O}_{\mathbb{P}^n}((1-d)H).$$

But $(1-d)H$ is a negative line bundle and

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H - M)) = H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((1-d)H)) = H^1(\mathbb{P}^n, \Omega^0_{\mathbb{P}^n}((1-d)H)) = 0$$

by Kodaira vanishing theorem since $1 + 0 < n$. Hence we see from the exact cohomology sequence that the restriction map r is surjective as required. (See Griffiths and Harris, pages 177+.)

Q-3) If you have to name only one theorem as the most important, or most exciting theorem among the ones we learned this semester in this course which one would you name? Explain why you prefer that particular theorem. No proofs are required for this answer. Just explain why that theorem seems most important or most exciting for you.

Solution-3 This is very personal. My candidates would be:

- *Hard Lefschetz Theorem*, page 122, GH.
- *Kodaira-Nakano Vanishing Theorem*, page 154, GH.
- *Theorem B*, page 159, GH.
- *Lefschetz Theorem on (1,1)-cycles*, page 161, GH.
- *Lefschetz Hyperplane Sections Theorem*, page 168, GH.
- *Kodaira Embedding Theorem*, page 181, GH.

Q-4) Summarize in your own word what you learned from James D. Lewis's talk at the ODTÜ-Bilkent Algebraic Geometry Seminar on 29 May 2020.

Solution-4

The video recording of James Lewis's talk is here:

Topic: MATH Seminar Room 102's Personal Meeting Room Start Time : May 29, 2020 03:51 PM (The first 57 seconds of the recording is bad but the rest of the recording is perfect, so don't get discouraged at the beginning!)

Meeting Recording: Zoom Cloud Link for Lewis' Talk

And the link for the pdf file of the talk is: Lewis' Talk pdf file

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