Math 431 ALGEBRAIC GEOMETRY
Homework 1 Solution Key

Exercise 4.3 page 17) Prove that there are no holomorphic differentials on the Riemann sphere except the trivial one.

Let ω be a holomorphic differential, fix a point $q \in S$ and consider the function

$$f(p) = \int_q^p \omega, \quad \text{for} \quad p \in S.$$

The path taken from q to p is not important since for any two such paths the integral over their sum is zero by Stokes' theorem (Theorem 4.6 page 17). Thus f is a well defined function which is holomorphic since it is the integral of a holomorphic form. This last observation can be checked directly: let $\omega = (u + iv)(dx + idy)$ and $f = U + iV$. Then you can easily verify Cauchy-Riemann equations for f assuming $u + iv$ satisfies them. Here we are of course using a local coordinate chart together with the information that S can be covered by two charts. Finally f being a holomorphic function on a compact space must be constant, so $df = 0$, but from its definition $df = \omega$. So $\omega = 0$.

Exercise 7.1 page 31) Verify that all automorphisms of $\mathbb{P}^1$ are of the form

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc \neq 0.$$

Since f is an automorphism it is one-to-one so it has only one pole and that is a simple pole. We have two cases:

Case-1) f is holomorphic on $\mathbb{C}$ with a simple pole at infinity. In this case f is entire, has a power series expansion around the origin with infinite radius of convergence. Substituting $1/t$ for z gives the power series expansion around $t = 0$ which corresponds to the expansion of f at infinity. Since f has a pole of order one at infinity, the new expansion must have a pole of order one at $t = 0$. This forces f to be linear, $f(z) = az + b$, where $a \neq 0$ since f cannot be constant.

Case-2) f has a simple pole at $z = \alpha$ and is holomorphic everywhere else including infinity. Then the Laurent expansion of f around α is of the form

$$f(z) = \frac{a_{-1}}{z - \alpha} + a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots$$

Since f is holomorphic at infinity, $f(1/t)$ must be holomorphic at $t = 0$. This forces $a_i = 0$ for all $i > 0$. So f is of the form

$$f(z) = \frac{a_0 z + (a_{-1} - a_0\alpha)}{z - \alpha}.$$

In both cases f is of the form

$$f(z) = \frac{az + b}{cz + d}$$

and since f is invertible we must have $ad - bc \neq 0$.
