Date: April 16, 2010, Friday NAME:..
Time: 08:40-10:30
Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

In this exam we are working on an algebraically closed field k.

Q-1) The J-invariant of a smooth plane cubic $y^2 = 4x^3 - ax - b$ is given by $J(a,b) = \dfrac{a^3}{a^3 - 27b^2}$. Explain what happens when $a^3 - 27b^2 = 0$? Prove your claims. *(Here clearly $k = \mathbb{C}$.)*

Solution: By direct calculation we can show that when the above curve is smooth, then $a^3 \neq 27b^2$. Start with the gradient $(12x^2 - a,\ -2y) = (0,\ 0)$ and incorporating the equations $x^2 = a/12$ and $0 = 4x^3 - ax - b$, you will find $\pm a^{3/2} = -3\sqrt{3}b$ and after squaring both sides you will find $a^3 = 27b^2$ being the condition for the curve to be singular.

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Q-2) Let H be the hyperplane in $\mathbb{P}^n$ given by $a_0x_0 + \cdots + a_nx_n = 0$. Give an explicit description of a map

$$\phi : \mathbb{P}^n \setminus H \to \mathbb{A}^n$$

which is an isomorphism. In other words, give formulas for $\phi([x_0 : \cdots : x_n])$ and for $\phi^{-1}(u_1, \ldots, u_n)$.
(Without loss of generality you may assume that $a_0 \neq 0$.)

Solution:

$$\phi([x_0 : \cdots : x_n]) = \left(\frac{x_1}{a_0x_0 + \cdots + a_nx_n}, \ldots, \frac{x_n}{a_0x_0 + \cdots + a_nx_n} \right)$$

and

$$\phi^{-1}(u_1, \ldots, u_n) = \left[\frac{1}{a_0}(1 - \sum_{i=1}^{n} a_iu_i) : u_1 : \cdots : u_n \right].$$

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Q-3) Let L be the line in $\mathbb{P}^2$ given by the equation $y = x$. What are the points at infinity of this line? Prove or disprove the statement that there exists a curve C in $\mathbb{P}^2$ given by the local coordinates $f(x, y) = 0$ where $\deg f \geq 1$ such that C has no point at infinity.

Solution: L is given by $[t : t : 1]$ for $t \in k$ and since $[t : t : 1] = [1 : 1 : 1/t]$ goes to $[1 : 1 : 0]$ as $t \to \infty$, the point at infinity is $[1 : 1 : 0]$.

Assume that there exists a non-trivial curve C which does not intersect the line H at infinity. By the previous problem $\mathbb{P}^2 \setminus H$ is isomorphic to $\mathbb{A}^2$ and thus C is isomorphic to an affine plane curve. Then since C is non-trivial, at least one of the coordinate functions in $\mathbb{A}^2$ gives a non-trivial global regular function on C. But on a projective variety there are no non-trivial global regular functions. This contradiction shows that no such non-trivial curve exists.

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Q-4) Find all singular points of the curve C which is given by $y^2 = x^3 - x^2$. Blow up the curve at each singularity and explicitly show how the singularity is resolved.

Solution:

By considering the Jacobian of the curve, we find that the only singularity is at the origin. Blowing the curve at the origin and mapping it to affine coordinates amounts to putting $x = X$, $y = XY$ for one chart, and $x = UV$, $y = U$ for the other chart. These give $Y^2 = X - 1$ and $1 = UV^3 - V^2$ both of which are smooth.

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Q-5) Let C and C' be two plane curves in $\mathbb{P}^2$ with $I(C) =< f >$ and $I(C') =< g >$. Let p be a point in $\mathbb{P}^2$.

Define the intersection multiplicity $I_p(C, C')$ of C and C' at p.

Also state the Bezout's theorem for the total intersection number $C \cdot C' := \sum_{p \in \mathbb{P}^2} I_p(C, C')$.

Calculate the intersection multiplicity of $y = x^2$ and $y = x^2 + 1$ at infinity, first by using the definition and then using Bezout's theorem if it is applicable in this case.

Solution:

$$I_p(C, C') := \dim_k \mathcal{O}_{\mathbb{P}^2,p} / < f, g > .$$

$$C \cdot C' := \sum_{p \in \mathbb{P}^2} I_p(C, C') = (\deg f)(\deg g).$$

The only intersection of these curves is at the point $[0 : 1 : 0]$. Since their degrees multiply to 4, the intersection multiplicity at this point must be 4. In fact if we consider the definition we see immediately that $z = x^2$ and $x^4 = 0$ are the relations. Hence $1, x, x^2, x^3$ form a basis.