Due Date: 27 May 2011, Friday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Homework 3

1	2	TOTAL
50	50	100

Please do not write anything inside the above boxes!

Check that there are 2 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Also note that** if you write down something which you don't believe yourself, the chances are that I will not believe it either.

NAME: STUDENT NO:

Q-1) Show that for every integer $g > 2$, there exist at least two compact Riemann surfaces X and Y such that both have genus g but one is hyperelliptic and the other is non-hyperelliptic.

Solution:

For this I simply refer you to

http://www.bilkent.edu.tr/ sertoz/courses/math431/2002/mm431hwk3.pdf

Another approach to the existence of non-hyperelliptic curves for every genus $g \geq 4$ is given by Exercise V.2.10 on page 385 of Hartshorne's *Algebraic Geometry*.

For a constructive approach see Walker's *Algebraic Curves*, page 189 Theorem 7.4.

Remark: The aim of this problem is to realize that the task of constructing non-hyperelliptic curves is highly non-trivial. A generic Riemann surface is non-hyperelliptic but to construct one such curve is difficult. Compare this situation to the abundance of transcendental numbers in the set of real numbers.

NAME: STUDENT NO:

Q-2) Show that a canonical curve of genus $g > 5$ is never a complete intersection.

Solution:

Assume C in $\mathbb{P}^{g-1}$ is a canonical curve of genus $g > 5$. By definition C is non-degenerate so does not lie in any hyperplane. Let H be a generic hypersurface such that $C \cap H$ consists of $\deg C = 2g - 2$ points.

Assume that C is a complete intersection. Then by dimension considerations $C = V_1 \cap \cdots \cap V_{g-2}$ where each V_i is a hypersurface of degree $d_i > 1$.

We will then observe that $H \cap V_1 \cap \cdots \cap V_{g-2}$ consists of $2g - 2$ points on one hand, since $\deg C = 2g - 2$, and $d_1 \cdots d_{g-2}$ points on the other hand by the generalized Bezout theorem.

But clearly we have

$$d_1 \cdots d_{g-2} \geq 2^{g-2} > 2g - 2 \text{ when } g > 5,$$

which gives a contradiction. Hence a canonical curve of genus $g > 5$ can never be a set theoretical complete intersection.

Remark: See also problem 5 on page 211, which is about genus 5 curves. Here we study the general case. For the generalized Bezout theorem you can refer to Griffiths-Harris's *Principles of Algebraic Geometry*, page 670, or Fulton's *Intersection Theory*, page 145, or follow the link http://mathoverflow.net/questions/42127/generalization-of-bezouts-theorem