Date: 20 March 2012, Tuesday
Time: 13:40-15:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam 1 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

We work over the complex numbers unless explicitly stated otherwise. $Z(P)$ means the zero set of the polynomial P.

This is an open book, open notebook exam.

NAME: STUDENT NO:

Q-1) Describe a way of parameterizing all integer solutions of the equation

$$x^2 + y^2 = z^2.$$

Why does your method fail to parametrize all integer solutions of

$$x^n + y^n = z^n \text{ for } n > 2.$$

What is known about the integer solutions of this last equation?

Solution:

One common method to generate the integral solutions of the equation $x^2 + y^2 = z^2$ is to treat the problem as generating rational solutions of the equation $s^2 + t^2 = 1$ where $s = x/z$ and $t = y/z$. For this purpose consider the unit circle $s^2 + t^2 = 1$ and the line $t = \lambda(s + 1)$ where λ is a rational number. The line and the circle intersect at the points $(-1, 0)$ and $(\frac{1-\lambda^2}{1+\lambda^2}, \frac{2\lambda}{1+\lambda^2})$. You can argue that this generates all the rational points on the circle. Then letting $\lambda = m/n$, where m and n are integers, you get the integral solutions of the original equation as

$$x = n^2 - m^2, y = 2mn, z = n^2 + m^2.$$

There are numerous reasons why this method does not work for higher powers. First of all the line which parameterizes the rational points will intersect the curve in many points and in general none of these points can be solved as we did above using the quadratic formula.

It is now known, through the work of Andrew Wiles in 1995, that there are no non-trivial integer solutions to the general equation as Fermat predicted in 1637.

NAME: STUDENT NO:

Q-2) Let $L \subset \mathbb{P}^2$ be aline and $C \subset \mathbb{P}^2$ be a nonsingular conic. Assume that the homogeneous coordinates $[x : y : z] \in \mathbb{P}^2$ are so chosen that L is given by $\alpha x + \beta y + \gamma z = 0$ and C is given by $yz = x^2$.

(i) Construct a homeomorphism $f : \mathbb{P}^1 \to L$.

(ii) Construct a homeomorphism $g : \mathbb{P}^1 \to C$.

(iii) Does there exist a projective transformation $T : \mathbb{P}^2 \to \mathbb{P}^2$ such that $T(L) = C$?

(Grading: 9+1+10 points.)

Solution:

A line in $\mathbb{P}^2$ is a plane in $\mathbb{C}^3$ passing through the origin. In this case this is the plane given by $\alpha x + \beta y + \gamma z = 0$. Take two linearly independent vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) in $\mathbb{C}^3$ satisfying $\alpha a_i + \beta b_i + \gamma c_i = 0$, $i = 1, 2$. Then L can be parametrized by

$$x = a_1 s + a_2 t, y = b_1 s + b_2 t, z = c_1 s + c_2 t \text{ where } [s, t] \in \mathbb{P}^1.$$

This also gives the required homeomorphism between $\mathbb{P}^1$ and L.

The homeomorphism between $\mathbb{P}^2$ and C is given by $[s : t] \to [st : t^2 : s^2]$ as shown in the book.

There cannot exist a projective transformation carrying L onto C. Any line intersects L in one point and C in two points by Bezout Theorem. These numbers would be preserved if there was a projective transformation between L and C.

NAME: STUDENT NO:

Q-3) Let $[x : y : z] \in \mathbb{RP}^2$ be the homogeneous coordinates of the real projective plane. The projective curve $x^3 + y^3 - 3xyz = 0$ is called the folium of Descartes.
Find the singularities of the folium and describe the nature of the singularities.
Let $(X, Y) = (x/z, y/z)$ and $(U, V) = (x/y, z/y)$ be Euclidean coordinates. The graph of the folium in XY-coordinates is given in Figure 1, and the graph in UV-coordinates is given in Figure 2. In Figure 1 the folium is given as the union of several arc together with their directions. Show where these arcs are in Figure 2.

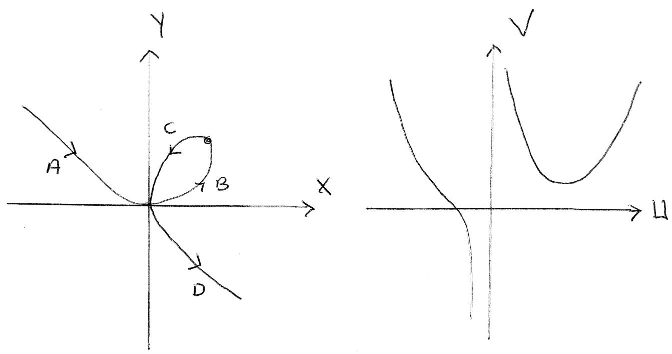
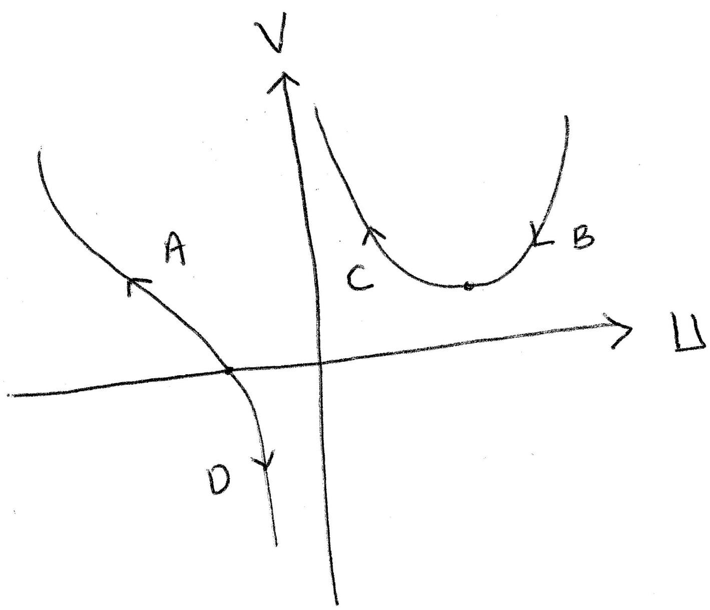


Fig 1 Fig 2

Solution:

The only singularity is at the point $[0 : 0 : 1]$ which is an ordinary singularity with two orthogonal tangent lines. The arcs are depicted in the following figure.



NAME: STUDENT NO:

Q-4) Let L and M be two projective lines in $\mathbb{P}^2$. Let p_1, p_2, p_3 be three distinct points on L, and let q_1, q_2, q_3 be three distinct points on M, none of these six points lying on $L \cap M$. Let L_{ij} be the line joining p_i to q_j, $i, j = 1, 2, 3$ and $i \neq j$. Finally let r_{ij} be the point of intersection of the lines L_{ij} and L_{ji}, $i, j = 1, 2, 3$ and $i < j$.
Show that the points r_{12}, r_{13}, r_{23} are collinear.

Solution:

Let C be the curve $L_{12} \cup L_{31} \cup L_{23}$, D be the curve $L_{32} \cup L_{21} \cup L_{13}$. Assume that C and D are given by the homogeneous cubic polynomials P and Q respectively. Let the lines L and M be given by the linear forms F_1 and F_2 respectively. Then the curve $E = L \cap M$ is given by the quadric $F_1 F_2$.

$C \cap D = \{p_1, p_2, p_3, q_1, q_2, q_3, r_{12}, r_{13}, r_{23}\}$. Six of these points lie on E. Pick another point $p \in E$ different than these six points. Let $F = Q(p)P - P(p)Q$. Then $F(p) = 0$ and moreover $F(p_i) = Q(p_i) = 0$ for $i = 1, 2, 3$. Then the cubic curve $Z(F)$ and the quadratic curve E have more than six points in common, so they must have common factor. Suppose F_1 is common. Let $F = F_1 G$ where G is quadratic. But $Z(G)$ and M have more than two points in common, so they must have a common component also. Hence F_2 divides G. Similarly if F_2 divides F.

We find that $F = F_1 F_2 F_3$, where F_3 is necessarily linear.

Now we conclude that the remaining points r_{12}, r_{13}, r_{23} lie on the line $Z(F_3)$.

NAME: STUDENT NO:

Q-5)

(i) Let C and D be nonsingular projective curves of degrees n and m in $\mathbb{P}^2$. Show that if C is homeomorphic to D, then either $n = m$ or $\{n, m\} = \{1, 2\}$.

(ii) Assuming that $[x : y : z]$ are the homogeneous coordinates in $\mathbb{P}^2$, let L be the line in $\mathbb{P}^2$ given by the linear polynomial $z = 0$. Consider the map

$$\phi : L \to \mathbb{P}^2$$

given by $\phi([s : t : 0]) = [st^3 : (s + t)^4 : t^4]$. Construct a homogeneous polynomial $P(x, y, z)$ such that $\phi(L) = Z(P)$ in $\mathbb{P}^2$. Show that ϕ is a homeomorphism between L and $Z(P)$. What is the degree of P? Does this contradict the conclusion of part (i)?

Solution:

The degree-genus formula gives the genus g in terms of the degree d as

$$g = \frac{1}{2}(d - 1)(d - 2).$$

Two curves are homeomorphic if and only if their genus are the same. This proves the first part.

For the second part observe that the required polynomial is $P(x, y, z) = yz^3 - (x + z)^4$. The curve $Z(P)$ has a singularity at $[0 : 1 : 0]$. This is a triple singularity. Any line through $[0 : 1 : 0]$ intersects $Z(P)$ at only one other point and this can be used to parametrize the curve $Z(P)$. Moreover, joining any point on the curve to the point $[0 : 1 : 0]$ and taking the slope of the line gives an inverse to ϕ. It then follows that $\phi(L) = Z(P)$ and ϕ is a homeomorphism. The degree of L is one and the degree of $Z(P)$ is four. This does not contradict part (i) since part (i) is about nonsingular curves whereas here the curve $Z(P)$ is not smooth.