Final Exam for Math 431 **Due Date: 20 May 2022**

1) In an appendix to *Introduction to Algebraic Curves*, Griffiths sketches a proof of the following statement.

> If C is a compact complex Riemann surface, then there exists an immersion of C into $\mathbb{P}^2$ such that $f(C)$ is an algebraic curve with at most double points as singularities.

Since he uses this result for his development of the theory, he has to prove this using elementary methods.

Prove this result now using whatever we learned so far about Riemann surfaces and algebraic geometry.

2) Let C be a smooth cubic curve in $\mathbb{P}^2$, the ground field being $\mathbb{C}$. For any $p, q \in C$, let L be the line through p and q when $p \neq q$, and be the tangent line to C at p when $p = q$. By Bezout's theorem we have $L \cdot C = p + q + r$ for some $r \in C$. This defines a map $\phi\colon C \times C \to C$ as $\phi(p, q) = r$, where r is defined as above. Fix a point $p_0 \in C$. Define $p \oplus q$ for any $p, q \in C$ as

$$p \oplus q = \phi(p_0, \phi(p, q)).$$

Show that:

(i) $p \oplus q = q \oplus p$ for any $p, q \in C$

(ii) $p_0 \oplus p = p$ for any $p \in C$.

(iii) For every $p \in C$ there exists a $q \in C$ such that $p \oplus q = p_0$.

(iv) $p \oplus (q \oplus r) = (p \oplus q) \oplus r$ for any $p, q, r \in C$.

Thus C is an abelian group under this operation.

Compose your solutions in a way to convince yourself, not me!

Write your solutions on A4 papers, put your name on the top of each paper and staple them together. Do not use other fancy covers!

Thank you.