Due Date: 1 November 2012, Thursday NAME:..

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Math 503 Complex Analysis – Exam 02

1	2	3	4	5	TOTAL
40	60	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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Q-1) Explain why the Method of Partial Fractions of Calculus works. (*In other words, in this method we write a particular rational function as a sum of fractions of certain forms with unknown coefficients and then we solve for these coefficients. Show that we can always solve for these coefficients with the required equality holding.*)

Solution:

There are two main points to observe here. The first one is the following.

Let $P_1(z), \ldots, P_n(z)$ be the principal parts of our rational function at its poles. Then the sum of these principal parts equals the rational function. (prove!)

The second point to observe is the reflection principal. If $f(z)$ denotes our rational function which is real on the real line, then $f(\bar{z}) = \overline{f(z)}$. This means that the coefficients of the principal part at a non-real pole a are the complex conjugates of the corresponding coefficients at $\bar{a}$. Summing up these two principal parts we obtain real coefficients.

Now you need to write in detail that a real partial fraction decomposition is always possible using the above observations.

Here are the details:

Let $f(z) = \dfrac{G(z)}{H(z)}$, where G and H are polynomials in z with real coefficients and $\deg G < \deg H$. Assume that $\deg H = n$ and let $a_1, \ldots, a_m$ be its distinct roots, not counting multiplicity. Here clearly $0 < m \leq n$. Without loss of generality we can assume that H is a monic polynomial.

Let $P_1(z), \ldots, P_m(z)$ be the principal parts of f at the roots $a_1, \ldots, a_m$, and consider the function $\phi(z) = f(z) - (P_1(z) + \cdots + p_m(z)$. Clearly ϕ is an entire function. Either $\phi(z)$ is identically zero or it is a non-constant rational function with the degree of the numerator strictly less than the degree of the denominator. But in that case $\lim_{z\to\infty} \phi(z) = 0$ which in turn implies that $\phi(z)$ is bounded and hence is constant on $\mathbb{C}$. This contradiction shows that $\phi(z)$ is identically zero, or equivalently

$$f(z) = P_1(z) + \cdots + P_m(z).$$

This is a partial fractions decomposition of f. We now show that a decomposition into fractions is possible with only real coefficients.

If a is a real root of $H(z)$ with multiplicity k, then $(z-a)^k f(z)$ is analytic at a and has a Taylor series at a given by

$$(z-a)^k f(z) = c_0 + c_1(z-a) + \cdots + c_r(z-a)^r + \cdots$$

Since $(z-a)^k f(z)$ is a real rational function, all the coefficients of the Taylor series are real. In particular, if $P(z)$ is the principal part of $f(z)$ at $z = a$, then

$$P(z) = \frac{c_0}{(z-a)^k} + \cdots + \frac{c_{k-1}}{z-a},$$

and has only real coefficients.

Next assume that a is a non-real root of $H(z)$ of multiplicity k. Since H is a real polynomial, $\bar{a}$ is also a root of the same multiplicity. Then $(z-a)^k f(z)$ and $(z-\bar{a})^k f(z)$ are analytic at $z = a$ and $z = \bar{a}$ respectively. Their Taylor series at a and $\bar{a}$ can be written as

$$(z-a)^k f(z) = c_0 + c_1(z-a) + \cdots + c_r(z-a)^r + \cdots \qquad (*)$$

and

$$(z-\bar{a})^k f(z) = b_0 + b_1(z-\bar{a}) + \cdots + b_r(z-\bar{a})^r + \cdots \qquad (**)$$

Taking complex conjugates of both sides in the first expansion ($*$), we find

$$(\bar{z}-\bar{a})^k \overline{f(z)} = \bar{c_0} + \bar{c_1}(\bar{z}-\bar{a}) + \cdots + \bar{c_r}(\bar{z}-\bar{a})^r + \cdots$$

From the reflection principal we have

$$(\bar{z}-\bar{a})^k \overline{f(z)} = (\bar{z}-\bar{a})^k f(\bar{z}),$$

and using the expansion ($**$), we see that

$$b_r = \overline{c_r},\ r = 1, 2, \ldots$$

Now let $P(z)$ and $Q(z)$ be the principal parts of $f(z)$ at $z = a$ and $z = \bar{a}$ respectively. We then have

$$P(z) = \frac{c_0}{(z-a)^k} + \cdots + \frac{c_{k-1}}{z-a},$$

and

$$Q(z) = \frac{\bar{c_0}}{(z-\bar{a})^k} + \cdots + \frac{\overline{c_{k-1}}}{z-\bar{a}}.$$

Let $b = -(a+\bar{a})$ and $c = |a|^2$. Then $(z-a)(z-\bar{a}) = z^2 + bz + c$ is a real polynomial, and we have

$$P(z) + Q(z) = \sum_{\ell=1}^{k} \left(\frac{c_{k-\ell}}{(z-a)^\ell} + \frac{\overline{c_{k-\ell}}}{(z-\bar{a})^\ell} \right) = \sum_{\ell=1}^{k} \frac{c_{k-\ell}(z-\bar{a})^\ell + \overline{c_{k-\ell}}(z-a)^\ell}{(z^2+bz+c)^\ell}$$

Note that the complex conjugate of the numerator $c_{k-\ell}(z-\bar{a})^\ell + \overline{c_{k-\ell}}(z-a)^\ell$ is equal to itself when z is considered as a real variable. Hence this numerator is a real polynomial of z of degree ℓ. Therefore for each $\ell = 1, \ldots, k$, we can find real coefficients $e_1, \ldots, e_s$ and $d_1, \ldots, d_s$, where $s = \lfloor \ell/2 \rfloor$ such that

$$c_{k-\ell}(z-\bar{a})^\ell + \overline{c_{k-\ell}}(z-a)^\ell = \sum_{i=1}^{s} \left(e_i z [p(z)]^i + d_i [p(z)]^i \right) + (E_\ell z + D_\ell),$$

for some real numbers e and d, where $p(z) = z^2 + bz + c$. It follows that for each ℓ,

$$\frac{c_{k-\ell}(z-\bar{a})^\ell + \overline{c_{k-\ell}}(z-a)^\ell}{p(z)^\ell} = \sum_{i=1}^{s} \frac{e_i z + d_i}{p(z)^{\ell-i}} + \frac{E_\ell z + D_\ell}{p(z)^\ell}.$$

It now follows by induction that

$$P(z) + Q(z) = \sum_{\ell=1}^{k} \frac{E_\ell z + D_\ell}{(z^2+bz+c)^\ell},$$

as required.

This completes the proof that the real coefficients required for the partial fraction decomposition exist as claimed.

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Q-2) Prove or disprove the following statement:
Let D be the open unit disk in the plane. For every positive integer n let $z_n = 1 - 1/n$. Then there exists a meromorphic function on D, whose only poles are $\{z_n\}_{n=1}^{\infty}$, and whose principal part at z_n is $\dfrac{n^2}{(z-z_n)^2} + \dfrac{n}{z-z_n}$, $n = 1, 2 \ldots.$

Solution:

I refer you to pages 77-78 of a master thesis written on Mittag-Leffler Theorem:

http://people.math.sfu.ca/ tarchi/turnermsc2007.pdf

To adopt the proof there to the solution of this problem, you need to do the following assignments:

Let $\Omega = D$, and $A = \{z_n\}_{n=1}^{\infty}$. Also let

$$P_{z_n} = \frac{n^2}{(z-z_n)^2} + \frac{n}{z-z_n}, \quad n = 1, 2 \ldots.$$

In the proof let K_n be the disk around the origin whose radius r_n satisfies $|z_{n-1}| < r < |z_n|$, $n = 1, 2, \ldots.$

Now the rest of the proof shows that a meromorphic function with the required properties exists.