Due Date: 9 November 2012, Friday NAME:...

Ali Sinan Sertöz STUDENT NO:...

Math 503 Complex Analysis – Exam 03

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) On page 309 we have:

$$\sum_{j=1}^{\infty} j|c_j|^2 r^{-2j} \le r^2 \text{ for all } r > 1. \tag{47.5}$$

The book concludes that

$$\sum_{j=1}^{\infty} j|c_j|^2 \le 1 \text{ by letting } r \to 1.$$

Either justify the steps involved in taking this limit or obtain $\sum_{j=1}^{\infty} j|c_j|^2 \le 1$ using equation (47.5) through your own arguments

Solution:

For this limit process to make sense, we must first show that $\sum_{j=1}^{\infty} j|c_j|^2$ converges.

Let N be any positive integer and choose any r with $1 < r < 2$. Then letting

$$G_N(r) := \sum_{j=1}^{N} j|c_j|^2 r^{-2j}$$

we have

$$G_N(r) < \sum_{j=1}^{\infty} j|c_j|^2 r^{-2j} \le r^2 < 4.$$

As r descents to 1, $G_N(r)$ increases and is always bounded, so has a limit, say $G_N < 4$.

The sequence G_N is increasing and is bounded by 4, so has a limit, say G.

We now want to show that $G \le 1$.

Let

$$F(r) = \sum_{j=1}^{\infty} j|c_j|^2 r^{-2j}, \text{ for } r \ge 1.$$

By Abel's theorem we know that

$$\lim_{r\to 1^-} F(r) = F(1).$$

Finally we can take limit of both sides in equation (47.5) as r descents to 1 to obtain $F(1) = G \le 1$.

Another Solution: This is taken from Jeffrey S. Rosenthal's thesis of 1989, page 3.

Assume that

$$\sum_{j=1}^{\infty} j|c_j|^2 > 1. \tag{$*$}$$

Then we can find a positive integer N and a real number $\alpha > 0$ such that

$$\sum_{j=1}^{N} j|c_j|^2 = 1 + \alpha.$$

Choose $r > 1$ such that

$$r^{-2N} > \frac{1+\alpha/2}{1+\alpha} \text{ and } r^2 < 1 + \alpha/4.$$

This is equivalent to choosing an r such that

$$1 < r < \min\{(1+\alpha/4)^{1/2},\ \left(\frac{1+\alpha}{1+\alpha/2}\right)^{1/(2N)}\},$$

which is possible. Now for this r we have

$$\begin{aligned}\sum_{j=1}^{\infty} j|c_j|^2 r^{-2j} &\geq \sum_{j=1}^{N} j|c_j|^2 r^{-2j} \\ &\geq r^{-2N}\sum_{j=1}^{N} j|c_j|^2 \\ &> \frac{1+\alpha/2}{1+\alpha}(1+\alpha) \\ &= 1+\alpha/2 \\ &> 1+\alpha/4 \\ &> r^2,\end{aligned}$$

contradicting the result in equation (47.5). Hence our assumption ($*$) is wrong and

$$\sum_{j=1}^{\infty} j|c_j|^2 \leq 1.$$

For Rosenthal's thesis see the link:

`http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.55.9923&rep=rep1&type=pdf`

NAME: STUDENT NO:

Q-2) Let $g(z) = z + c_0 + \dfrac{c_1}{z} + \dfrac{c_2}{z^2} + \cdots$, for $|z| > 1$, be analytic and one-to-one. Show that if $|c_1| = 1$, then $g(z) = z + c_0 + \beta/z$ where $\beta \in \mathbb{C}$ with $|\beta| = 1$.

Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, for $|z| < 1$, be analytic and one-to-one. Assume that $|a_2| = 2$. Show that

$$f(z) = \frac{z}{(1-\lambda z)^2}, \text{ where } |\lambda| = 1.$$

Solution:

From the inequality

$$\sum_{j=1}^{\infty} j|c_j|^2 \leq 1,$$

it follows that if $|c_1| = 1$, then $c_n = 0$ for $n > 1$. This proves the first part.

Now apply this to the function

$$\frac{1}{F_1\left(\frac{1}{z}\right)} = \frac{1}{\frac{1}{z}\left[1 + \frac{1}{2}a_2\frac{1}{z^2} + \cdots\right]} = z - \frac{1}{2}a_2\frac{1}{z} + \cdots$$

on page 310. Since $|c_1| = |a_2/2| = 1$, from the first part it follows that

$$F_1(z) = \frac{z}{1-\lambda z^2}.$$

Since by definition we have

$$F_1(z) = z\sqrt{\frac{f(z^2)}{z^2}},$$

we conclude that

$$f(z) = \frac{z}{(1-\lambda z)^2}, \text{ where } |\lambda| = 1.$$