Due Date: 28 December 2012, Friday NAME:...

Please leave your homework in my mailbox until 17:30.

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Exam 11

1	2	3	4	5	TOTAL
100	0	0	0	0	100

Please do not write anything inside the above boxes!

Check that there is **1** question on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Let $G = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, and let $F(z)$ be analytic on G satisfying the following conditions.

i. $F(z+1) = zF(z)$ for $z \in G$.

ii. $|F(z)|$ is bounded for $1 \leq \operatorname{Re} z \leq 2$.

iii. $F(1) = 1$.

Show that $F(z) = \Gamma(z)$ for all $z \in \mathbb{C}$, where Γ is the complex Gamma function.

Solution: First note that, using similar arguments as for the Gamma function, $F(z)$ can be extended as a meromorphic function to all of the complex plane with simple poles at the non-positive integers with residues equal to $\dfrac{(-1)^n}{n!}$ at $z = -n$ for $n \in \mathbb{N}$.

Define the function

$$\phi(z) = F(z) - \Gamma(z), \quad z \in \mathbb{C}.$$

Clearly ϕ is now an entire function.

Next recall that $|\Gamma(z)|$ is bounded in every finite strip $0 < a \leq \operatorname{Re} z \leq b$. Using this, we conclude that $|\phi(z)|$ is bounded on the strip $1 \leq \operatorname{Re} z \leq 2$. Now using the functional equation $\phi(z+1) = z\phi(z)$, and the fact that $\phi(1) = 0$, we see that $|\phi(z)|$ is bounded on the strip $0 \leq \operatorname{Re} z \leq 1$.

Finally define a new function

$$g(z) = \phi(z)\phi(1-z), \quad z \in \mathbb{C}.$$

Clearly, $g(z)$ is analytic and is bounded on the strip $0 \leq \operatorname{Re} z \leq 1$. Using the functional equation for ϕ twice, we get easily that

$$g(z+1) = -g(z),$$

which implies both that $|g(z)|$ is bounded in the strip $0 \leq \operatorname{Re} z \leq 2$, and that it is periodic of period 2;

$$g(z+2) = -g(z+1) = g(z).$$

Hence by Liouville's theorem $g(z)$ is constant. But since $\phi(1) = 0$, we must also have $g(0) = 0$, and hence $g(z) \equiv 0$. This forces $\phi(z) \equiv 0$, which in turn gives

$$F(z) = \Gamma(z),$$

as claimed.