Due Date: 24 November 2014, Monday – Class time NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Homework 3 – Solutions

1	2	3	4	5	TOTAL
50	25	25	0	0	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) For any $a \in D = \{|z| < 1\}$, we define

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}, \quad \text{for} \quad z \in D.$$

We know that $\phi_a(D) = D$. Show that for any $a, b \in D$, there exists $c \in D$ such that

$$\phi_a \circ \phi_b = \lambda \phi_c,$$

where λ is a complex number with $|\lambda| = 1$. (Make sure to check that $|c| < 1$.)

Moreover let $\alpha \in \partial D$, i.e. $|\alpha| = 1$. Show that there exist $d \in D$ and $\beta \in \partial D$ such that

$$\phi_a(\alpha\phi_b(z)) = \beta\phi_d(z) \quad \text{for all} \quad z \in D.$$

(Again check that $|d| < 1$ and $|\beta| = 1$.)

Solution:

Straightforward calculation gives $c = \dfrac{a+b}{1+a\bar{b}} = \phi_{-b}(a) \in D$, and $\lambda = \dfrac{1+a\bar{b}}{1+\bar{a}b}$.

For the second part, again straightforward calculation gives $\beta = \dfrac{\alpha + a\bar{b}}{1+\alpha\bar{a}b} = \phi_{-a\bar{b}}(\alpha) \in \partial D$, and $d = \bar{\alpha}\,\dfrac{a+\alpha b}{1+\bar{\alpha}\bar{b}a} = \bar{\alpha}\,\phi_{-\alpha b}(a) \in D$.

NAME: STUDENT NO:

Q-2) [Conway, p133, Exercise 7] Suppose that f is analytic in a region containing the closure of $D = \{|z| < 1\}$. Assume that $|f(z)| < 1$ for $z \in D$. Assume further that f has a simple zero at $\frac{1}{4}((1+i)$ and a double zero at $\frac{1}{2}$. Can $f(0) = \frac{1}{2}$?

Solution:

Since $|\frac{1}{4}(1+i)| = \frac{\sqrt{2}}{4} \approx 0.35 < 1$, and since f maps this point to 0, we conclude that f maps D to D. Hence we have

$$f(z) = (z - \frac{1}{4}(1+i))(z - \frac{1}{2})^2 h(z),$$

for some analytic function h whose domain includes D. The data about f implies that h does not vanish at either $\frac{1}{4}(1+i)$ or at $\frac{1}{2}$. For $|z| = 1$, we have $|f(z)| = 1$, so it follows that

$$|h(z)| = \frac{1}{|z - \frac{1}{4}(1+i)|\,|z - \frac{1}{2}|^2}, \quad \text{for} \quad |z| = 1.$$

We have to find the minimum value of $|z - \frac{1}{4}(1+i)|\,|z - \frac{1}{2}|^2$ when $z = e^{i\theta}$. Rewriting this product as

$$P(\theta) = \sqrt{(\cos\theta - \frac{1}{4})^2 + (\sin\theta - \frac{1}{4})^2}\left((\cos\theta - \frac{1}{2})^2 + \sin^2\theta\right), \quad \text{for} \quad \theta \in [0, 2\pi),$$

we find that

$$P'(\theta) = -\frac{1}{4}\ \frac{-23\ \sin(t) + 12\ \sin(t)\cos(t) + 8 - 12\ (\cos(t))^2 + 5\ \cos(t)}{\sqrt{-8\ \cos(t) + 18 - 8\ \sin(t)}}.$$

We find that $P'(\theta) = 0$ when $\theta_1 = 0.09896934220$ or when $\theta_2 = 3.380102351$. It follows that

$$P(\theta_1) = 0.1937932780 < 0.194 \quad \text{and} \quad P(\theta_2) = 2.921310938.$$

Hence we have that $P(\theta) > 0.192$ for all $z = e^{i\theta}$. This gives

$$|h(z) < \frac{1}{0.192} \quad \text{for} \quad z = e^{i\theta}.$$

By the maximum modulus principle, this upper bound also bounds $|h(z)|$ for all $z \in D$. Finally we have

$$|f(0)| = |\frac{1}{4}(1+i)|\,| - \frac{1}{2}|^2\,|h(z)| < \frac{\sqrt{2}}{4}\,\frac{1}{4}\,\frac{1}{0.192} \approx 0.46 < \frac{1}{2}.$$

Hence $f(0)$ cannot be $1/2$.

NAME: STUDENT NO:

Q-3) [Conway, p133, Exercise 8] Is there an analytic function f on $D = \{|z| < 1\}$ such that $|f(z)| < 1$ for $|z| < 1$, $f(0) = \frac{1}{2}$, and $f'(0) = \frac{3}{4}$? If so, find such an f. Is it unique?

Solution:

If $f(a) = \alpha$, then $|f'(a)| \leq \dfrac{1 - |\alpha|^2}{1 - |a|^2}$, see Conway page 132, equation 2.3. Here $a = 0$ and $\alpha = 1/2$. Putting these values into the above inequality, we see that we must have

$$|f'(0)| \leq \frac{3}{4}.$$

We already have $f'(0) = 3/4$ so equality holds. In that case, there exists a number c with $|c| = 1$ such that $f(z) = \phi_{-\alpha}(c\phi_a(z))$, see Conway page 132, equation 2.4. Here $a = 0$ and $\alpha = 1/2$. This gives

$$f(z) = \frac{cz + (1/2)}{1 + (1/2)cz} = \frac{2cz + 1}{2 + cz}.$$

It follows that

$$f'(z) = \frac{3c}{(2 + cz)^2}.$$

Now we see that $f'(0) = 3c/2$. But it is given that $f'(0) = 3/4$, which forces $c = 1$. Thus such a function exists and is unique. In fact

$$f(z) = \frac{2z + 1}{z + 2}.$$