Due Date: 27 October 2016, Thursday
Class Time

NAME:..

STUDENT NO:.......................................

Math 503 Complex Analysis - Midterm Exam 1

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

NAME: STUDENT NO: DEPARTMENT:

Q-1) Let a, b, c, d be real numbers and let

$$A(a,b,c,d) = (a+ib)^{c+id} = U(a,b,c,d) + iV(a,b,c,d),$$

where U, V are the real and imaginary parts, and we calculate the principal value. Write explicit real formulas for U and V. Then using a software of your choice (WolframAlpha on the net, for example) evaluate the following using your formula and check that you get the correct values:

(i) $A(1,2,3,4)$

(ii) $A(7,0,2,0)$

(iii) $A(0,-7,-2,3)$

(iv) $A(0,7,2,3)$

Solution:

Define for $a \neq 0$

$$\begin{aligned} U(a,b,c,d) &= e^{(c/2)\,\ln\left(a^2+b^2\right) - d\arctan\left(\frac{b}{a}\right)} \cos\left(c\arctan\left(\frac{b}{a}\right) + (d/2)\,\ln\left(a^2+b^2\right)\right) \\ V(a,b,c,d) &= e^{(c/2)\,\ln\left(a^2+b^2\right) - d\arctan\left(\frac{b}{a}\right)} \sin\left(c\arctan\left(\frac{b}{a}\right) + (d/2)\,\ln\left(a^2+b^2\right)\right) \end{aligned}$$

Here $\arctan(b/a)$ is replaced by $\operatorname{sign}(b)(\pi/2)$ when $a = 0$.

We then have the following values:

$$\begin{aligned} A(1,2,3,4) &= 0.1290095939 + 0.03392409283\, i \\ A(7,0,2,0) &= 49 \\ A(0,-7,-2,3) &= -2.0500981220 + 0.97884311930\, i \\ A(0,7,2,3) &= -0.3972260723 + 0.1896601940\, i \end{aligned}$$

NAME: STUDENT NO: DEPARTMENT:

Q-2) Give a description of the Riemann surface of the mapping $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$, using colors when possible.

Solution:

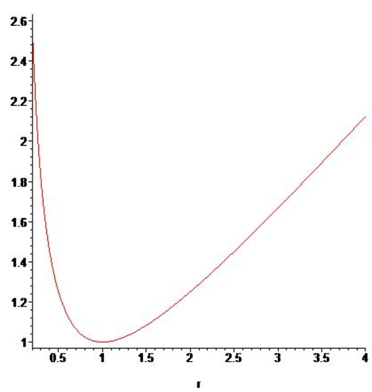
We write $z = re^{i\theta}$ for this function. If we set $w = u + iv$, then we have

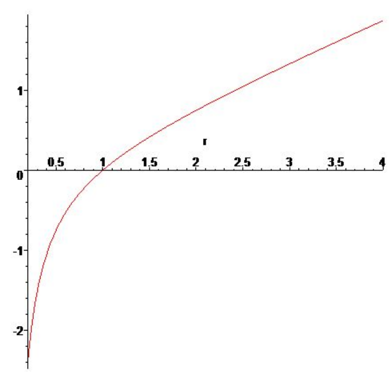
$$u(r,\theta) = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta,$$
$$v(r,\theta) = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta.$$

For notational convenience define

$$\alpha(r) = \frac{1}{2}\left(r + \frac{1}{r}\right) \quad \text{and} \quad \beta(r) = \frac{1}{2}\left(r - \frac{1}{r}\right).$$

The grpahs of $\alpha(r)$ and $\beta(r)$ are as follows.





When r is fixed, ignoring the extreme cases, we have

$$\frac{u^2}{\alpha(r)^2} + \frac{v^2}{\beta(r)^2} = 1.$$

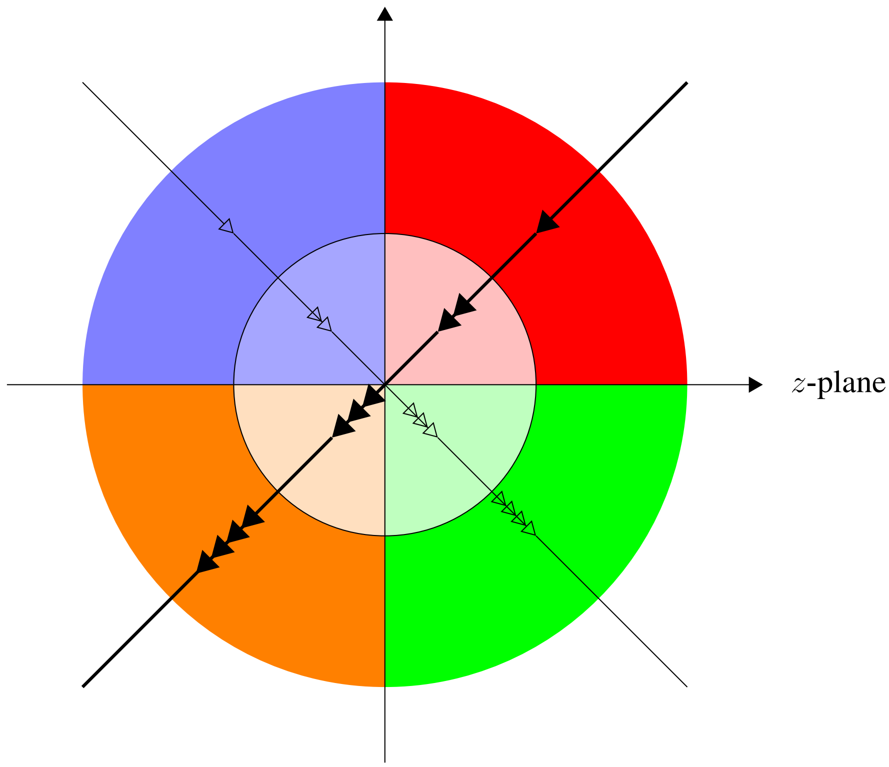
When we fix $r > 1$, circles in z-plane map to ellipses in w-plane with the same orientation around the origin. For $0 < r < 1$, the ellipses reverse their orientation. When $r = 1$, the unit circle maps to the inteval $[-1, 1]$ twice. Therefore we take two copies of w-plane, cut them along the interval $[-1, 1]$ and glue them along this interval in a cris-cross manner. Let us call the copy on top sheet I, and the one below sheet II. Circles in z-plane with radius $r > 1$ are mapped to sheet I, and circles with $0 < r < 1$ are mapped to sheet II.

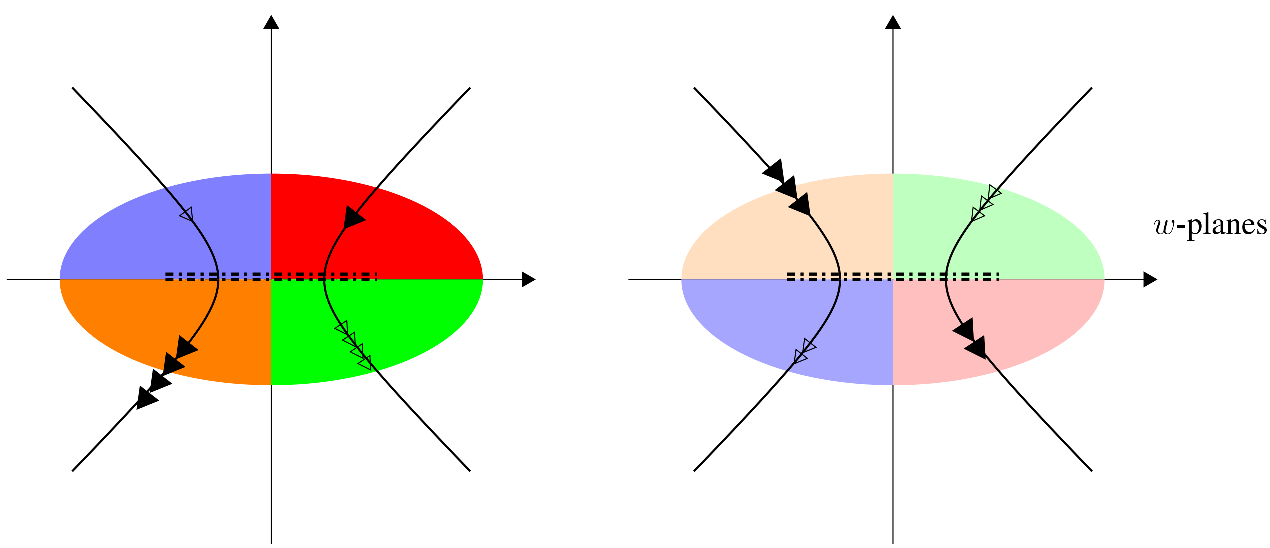
To check that our gluing is correct, consider ray in z-plane. If θ is fixed a generic ray goes to a hyperbola in $w$plane:

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1.$$

Following a point on a ray, we notice that the part of the ray within the unit disc is mapped to sheet II, and the part with $r > 1$ is mapped to sheet II.

You can follow the gluing from the following color codes.





NAME: STUDENT NO: DEPARTMENT:

Q-3 Every 2×2-matrix A with complex entries defines a Möbius transformation when $\det A \neq 0$, and conversely via the association

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \frac{az+b}{cz+d}$$

Here we consider Möbius transformations $T(z) = \dfrac{az+b}{cz+d}$ with $ad-bc = 1$. Define $\alpha(T) = (a+d)$. Prove the following.

1. If two Möbius transformations S and T are conjugate, i.e. there is another Möbius transformation U such that $USU^{-1} = T$, then $\alpha(T)^2 = \alpha(S)^2$. Can we say $\alpha(T) = \alpha(S)$?
2. A Möbius transformation T has exactly one fixed point if and only if $\alpha(T)^2 = 4$.
3. If $\alpha(T)^2 = 4$, then T is conjugate to a translation of the form $z \mapsto z + b$.

Solution:

1) We know that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ for two square matrices A and B. Therefore $\operatorname{trace} USU^{-1} = \operatorname{trace} U^{-1}US = \operatorname{trace} S$. But S and $-S$ both define the same Möbius transformation and $\operatorname{trace}(-S) = -\operatorname{trace} S$. So trace is defined up to a sign and we need to take squares. So $\alpha(S)^2 = \alpha(T)^2$.

2) If we set $T(z) = z$ with $ad - bc = 1$, then the discriminant of the resulting quadratic is $\Delta = (a+d)^2 - 4$. So T will have only one fixed point if and only if $\alpha(T)^2 = 4$.

3) This is classical so we follow the more or less standard approach.

First assume that T is not identity. Since $\alpha(T)^2 = 4$, by the second part above, T fixes only one point. Say $T(z_0) = z_0$. If $z_0 \neq \infty$, then let $g_1(z) = 1/(z - z_0)$. Then $g_1 \circ T \circ g_1^{-1}$ fixes only ∞ and is therefore of the form $z \mapsto z + b$. If $z_0 = \infty$, take $g_1 = id$. This shows that T is conjugate to a translation.

If T is identity, then we still have $\alpha(T)^2 = 4$, but we prefer not to call the identity map a translation by zero even though it is!

NAME: STUDENT NO: DEPARTMENT:

Q-4) Let γ be a closed simple rectifiable curve with the origin lying on its interior. Use the fundamental theorem of Calculus to evaluate

$$\int_\gamma \frac{1}{z}.$$

Solution:

An antiderivative for $1/z$ is the logarithm function $\log z$ with the branch cut along the non-negative real line. The curve γ cuts the positive real line at a point $r > 0$. Suppose γ moves around the origin counterclockwise. Then parametrize γ such that $\gamma(0) = \gamma(1) = r$. But the logarithm with this branch sees these as two different points: $\gamma(0) = re^{0i}$, $\gamma(1) = re^{2\pi i}$. Therefore by the fundamental theorem of algebra

$$\int_\gamma \frac{1}{z} = \left(\log z \Big|_{\gamma(0)}^{\gamma(1)} \right) = \left(\log z \Big|_{re^{0i}}^{re^{2\pi i}} \right) = 2\pi i.$$