Bilkent University

Final Exam
Math 503 Complex Analysis I
Due: 31 December 2020
Instructor: Ali Sinan Sertöz
Solution Key

You may and in fact must discuss your solutions with your friends. Consult any source available. After you ***understand*** the solution start writing your own solution in your own words. When you get stuck, again talk with your friends and consult sources before you continue to write your own solution in your own words.

At the end of each solution quote the sources that you used, including your friends's names who provided you with useful ideas. This is professionalism!

Never borrow your friend's written solution and never lend your written solution.

Then scan and save your solutions as one pdf file and mail it to me before the deadline.

Q-1) Let $f(z) = u(x, y) + iv(x, y)$ be an entire function. Prove or disprove the following.

(a) If there exist $w \in \mathbb{C}$ and $r > 0$ such that there is no $z \in \mathbb{C}$ such that $f(z) \in B(w; r)$, then f is constant.

(b) If there exists $M > 0$ such that $|u(x, y)| \leq M$ for all $z = x + iy \in \mathbb{C}$, then f is constant.

(c) If there exists $M > 0$ such that $|v(x, y)| \leq M$ for all $z = x + iy \in \mathbb{C}$, then f is constant.

Answer-1

(a) Suppose such a w and $r > 0$ exist. Then $|f(z)-w| \geq r$ for any $z \in \mathbb{C}$, and the function $\dfrac{1}{f(z) - w}$ is a bounded entire function which according to Liouville's theorem must be constant. Hence f itself is constant.

If you want to use a heavier machinery then you can quote Picard's Little Theorem (Conway, page 297) which says that an entire function which omits two values is constant. Here f omits a whole open disc and is certainly constant.

(b) Take $w = M + 1$ and $r = 1$, then apply part (a). Hence f is constant.

You can also show that since $e^{f(z)}$ is bounded and therefore constant, it forces f to be constant as well.

(c) Similar to part (b), take $w = i(M + 1)$ and $r = 1$. Hence f is constant.

Similarly here you can use $e^{-if(z)}$ which will be bounded and hence constant.

. . . .

Q-2) Prove or disprove the following.

(a) $f(z) = z$ is the only entire function with the propert that $f(n) = n$ for every integer n.

(b) $f(z) = \pi$ is the only entire function with the property that $f(n) = \pi$ for every integer n.

Answer-2

(a) This is not true. Let $g(z)$ be an entire function whose zero set is the set of integers. Such functions exist thanks to Weierstrass factorization theorem. Then $f(z) = g(z) + z$ is another entire function with the required property. In fact you can take $g(z) = \sin 2\pi z$.

(b) This is not true either since as above we can consider the function $\pi + \sin 2\pi z$.

. . . .

Q-3) Show that for every integer $n \geq 2$, we have

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n \sin \frac{\pi}{n}}.$$

Answer-3

Let $f(z) = \dfrac{1}{1+z^n}$ and consider the path $L_R = A_R + B_R + C_R$ where A_R is the path along the x-axis from 0 to $R > 1$, B_R is the circular arc from R to $R\exp(2\pi i/n)$ on the circle of radius R and center 0, C_R is the path along the line from $R\exp(2\pi i/n)$ to 0. We have

$$I_R := \int_{L_R} f(z)\, dz = 2\pi i \operatorname{Res}\left[f(z), z = \exp(\pi i/n)\right] = \frac{2\pi}{n}\left(\sin\frac{\pi}{n} - i\cos\frac{\pi}{n}\right).$$

We also have

$$\int_{A_R+C_R} f(z)\, dz = (1 - \exp(2\pi i/n)) \int_0^R \frac{dx}{1+x^n}.$$

Taking limits as $R \to \infty$, and noting that in that case the integral over B_R also goes to zero, we get

$$(1 - \exp(2\pi i/n)) \int_0^\infty \frac{dx}{1+x^n} = \frac{2\pi}{n}\left(\sin\frac{\pi}{n} - i\cos\frac{\pi}{n}\right).$$

Now equating the imaginary parts of both sides and simplifying gives us the required formula.

Q-4) Show that $\cos \pi z = \displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)$.

Answer-4

We reduce this back to an expression using the $\sin \pi z$ function as follows.

$$
\begin{aligned}
\cos \pi z &= \frac{\sin 2\pi z}{2 \sin \pi z} \\
&= \frac{\cancel{2\pi z} \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{n^2}\right)}{\cancel{2\pi z} \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)} \\
&= \frac{\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)\cancel{\left(1 - \frac{4z^2}{(2n)^2}\right)}}{\cancel{\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)}} \\
&= \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right).
\end{aligned}
$$

Q-5) Find a formula for the value of the integrals $I_n = \displaystyle\int_0^{\pi/2} \frac{dx}{1 + \tan^{n!} x}$, where $n \geq 4$ is an integer.

Answer-5

When $n \geq 4$, then $n! = 4m$ for some integer $m \geq 1$.

First use a real substitution $t = \tan x$.

$$
I_{4m} = \int_0^{\pi/2} \frac{dx}{1 + \tan^{4m} x} = \int_0^{\infty} \frac{dt}{(1+t^2)(1+t^{4m})}.
$$

Now let

$$
f(z) = \frac{1}{(1+z^2)(1+z^{4m})}.
$$

Let γ_R be the contour consisting of the real interval $[-R, R]$ together with the semicircle C_R of radius R centered at the origin, in the upper half plane and traversed in the positive direction, counterclockwise. When $R > 1$ we have

$$
\int_{\gamma_R} f(z)\, dz = (2\pi i)(\text{ sum of the residues of } f(z) \text{ in the upper half plane}).
$$

We also observe that the right hand side is independent of $R > 1$, but the left hand side depends on R. Taking limits as R goes to infinity, we get

$$
\lim_{R\to\infty} \int_{\gamma_R} f = \lim_{R\to\infty} \int_{C_R} f + \lim_{R\to\infty} \int_{[-R,R]} f = \lim_{R\to\infty} \int_{[-R,R]} f = 2I_{4m},
$$

since the limit of the integral on C_R is zero. We then have

$$
I_{4m} = \pi i(\text{ sum of the residues of } f(z) \text{ in the upper half plane}).
$$

We now recall that if $f(z) = \dfrac{p(z)}{q(z)}$, where p and q are analytic functions, a is a simple zero of q, and $p(a) \neq 0$, then $\mathrm{Res}(f(z), z = a) = \dfrac{p(a)}{q'(a)}$. First let

$$f(z) = \frac{p(z)}{q(z)}, \text{ where } p(z) = \frac{1}{(z+i)(z^{4m}+1)}, \text{ and } q(z) = z - i.$$

Then

$$\mathrm{Res}(f(z), z = i) = p(i) = \frac{1}{4i}.$$

Next let

$$f(z) = \frac{p(z)}{q(z)}, \text{ where } p(z) = \frac{1}{(1+z^2)}, \text{ and } q(z) = 1 + z^{4m}.$$

Let $z_0, z_1, \dots z_{2m-1}$ be the roots of $q(z) = 0$ in the upper half plane, where

$$z_k = \exp(\frac{2k+1}{4m}\,\pi i), \quad k = 0, \dots, 2m-1.$$

Observe that

$$z_{2m-k-1} = -\bar{z}_k, \quad k = 0, \dots, m-1.$$

Now a straighforward calculation shows that

$$\mathrm{Res}(f(z), z = z_k) + \mathrm{Res}(f(z), z = z_{2m-k-1}) = 0,$$

and hence

$$\sum_{k=0}^{2m-1} \mathrm{Res}(f(z), z = z_k) = 0.$$

Thus the sum of residues of $f(z)$ in the upper half plane is just $1/(4i)$ which was calculated above.

Thus finally we get

$$I_n = \int_0^{\pi/2} \frac{dx}{1 + \tan^{n!} x} = \frac{\pi}{4}, \quad n \geq 4.$$

In fact you can prove that

$$☺ \quad \int_0^{\pi/2} \frac{dx}{1 + \tan^{\alpha} x} = \frac{\pi}{4}, \quad \alpha \in \mathbb{R}. \quad ☺$$

Here is how to do it. Let

$$\phi(\alpha) = \int_0^{\pi/2} \frac{dx}{1 + \tan^{\alpha} x}.$$

We now show that $\phi'(\alpha) = 0$ as follows.

$$\begin{aligned}
\phi'(\alpha) &= \frac{d}{d\alpha}\int_0^{\pi/2} \frac{dx}{1+\tan^\alpha x} \\
&= \frac{d}{d\alpha}\int_0^{\infty} \frac{dt}{(1+t^2)(1+t^\alpha)} \quad \textit{(after putting } t = \tan x\textit{)} \\
&= \int_0^{\infty} \frac{d}{d\alpha}\frac{dt}{(1+t^2)(1+t^\alpha)} \quad \textit{(Leibniz integral rule)} \\
&= \int_0^{\infty} \frac{-t^\alpha \ln t\; dt}{(1+t^2)(1+t^\alpha)^2} \\
&= \int_0^{1} \frac{-t^\alpha \ln t\; dt}{(1+t^2)(1+t^\alpha)^2} + \int_1^{\infty} \frac{-t^\alpha \ln t\; dt}{(1+t^2)(1+t^\alpha)^2} \\
&= \int_1^{\infty} \frac{x^\alpha \ln x\; dx}{(1+x^2)(1+x^\alpha)^2} + \int_1^{\infty} \frac{-t^\alpha \ln t\; dt}{(1+t^2)(1+t^\alpha)^2} \quad \textit{(after putting } t = 1/x\textit{)} \\
&= 0,
\end{aligned}$$

as claimed. Now that we know that $\phi(\alpha)$ is constant, we calculate ϕ for an easy α.

$$\phi(0) = \int_0^{\pi/2} \frac{dx}{2} = \frac{\pi}{4}.$$

Hence $\phi(\alpha) = \frac{\pi}{4}$ for any $\alpha \in \mathbb{R}$.

Note added after I read your solutions: There is a much easier and neater way of getting this result. Here is how some of you did it.

Let

$$I_\alpha = \int_0^{\pi/2} \frac{dx}{1+\tan^\alpha x}. \tag{$*$}$$

Now put $x = \frac{\pi}{2} - t$. Then note that $\tan x = \frac{1}{\tan t}$. With this substitution we get

$$I_\alpha = \int_0^{\pi/2} \frac{\tan^\alpha t}{1+\tan^\alpha t}\; dt. \tag{$**$}$$

Finally adding $(*)$ and $(**)$ side by side we get

$$2I_\alpha = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

which then clearly shows that

$$I_\alpha = \frac{\pi}{4}.$$