Bilkent University

Homework # 05
Math 503 Complex Analysis I
Due: 18 December 2020 Friday
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Solution Key

Q-1) Show that $\Gamma(z)$ never vanishes.

Q-2) Show that

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}, \quad \text{for} \quad \operatorname{Re} z > 1,$$

where $\zeta(z)$ is the Riemann zeta function, and $\Lambda(n)$ is the Mangoldt function defined on positive integers as $\Lambda(n) = \log p$ if n is a power of the prime p, and is zero otherwise.

Answer-1:

Here we recall the definition of the Gamma function.

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+\frac{z}{n}}.$$

Also from the functional equation $\Gamma(1+z) = z\Gamma(z)$ we get $\Gamma(1-z) = -z\Gamma(-z)$. We now have:

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= -z\Gamma(z)\Gamma(-z) \\
&= -z \cdot \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+\frac{z}{n}} \cdot \frac{e^{\gamma z}}{-z} \prod_{n=1}^{\infty} \frac{e^{-z/n}}{1-\frac{z}{n}} \\
&= \frac{1}{z} \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} \cdot \frac{\pi}{\pi} \\
&= \frac{\pi}{\sin \pi z}.
\end{aligned}$$

Thus we proved that

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z.$$

The right hand side never vanishes, so the left hand side and hence $\Gamma(z)$ never vanishes.

Answer-2:

Let $n > 1$ be an integer such that $n = p^k m$ where p is prime and $(p, m) = 1$. Consider the product

$$\begin{aligned}
&\left(\frac{\log p}{p^z} + \frac{\log p}{(p^2)^z} + \cdots + \frac{\log p}{(p^k)^z}\right)\left(\frac{1}{(m)^z} + \frac{1}{(pm)^z} + \cdots + \frac{1}{(p^{k-1}m)^z}\right) \\
=&\frac{k \log p}{n^z} + \text{terms with denominator } r^z \text{ with } r \neq n
\end{aligned}$$

We recall the Von Mangoldt function defined on positive integers

$$\Lambda(n) = \begin{cases} \log p & n = p^k \text{ for some prime } p \text{ and some integer } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The above calcultion showed us that for $\operatorname{Re} z > 1$,

$$\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^z}\right) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z},$$

where

$$f(n) = k_1 \log p_1 + \cdots + k_\ell \log p_\ell = \log n, \quad \text{where } n = p_1^{k_1} \cdots p_\ell^{k_\ell} \text{ is the prime factorization of } n,$$

Since

$$\zeta'(z) = -\sum_{n=1}^{\infty} \frac{\log n}{n^z},$$

we just proved that

$$\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}\right)\zeta(z) = -\zeta'(z),$$

which proves the identity we want.