Due Date: 15 March 2013, Friday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 504 Complex Analysis II – Take-Home Exam 02 – Solutions

1	2	3	4	5	TOTAL
25	25	25	25	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.

NAME: STUDENT NO:

Q-1) Find the type and multiplier of each of the following transformations:

$$\text{(i)}\ \frac{z+1}{z+3}, \quad \text{(ii)}\ \frac{iz+1}{z+3i}, \quad \text{(iii)}\ iz+1, \quad \text{(iv)}\ \frac{-z}{z+4}.$$

[page 53, Exercise 2B]

Solution:

For a general Mobius transformation $T(z) = \dfrac{az+b}{cz+d}$, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ and } B = \begin{pmatrix} \frac{a}{\sqrt{\det A}} & \frac{b}{\sqrt{\det A}} \\ \frac{c}{\sqrt{\det A}} & \frac{d}{\sqrt{\det A}} \end{pmatrix}.$$

(i) $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, $\det A = 2$, $\sqrt{\det A} = \sqrt{2}$, $B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 3/\sqrt{2} \end{pmatrix}$, $\operatorname{tr} B = 2\sqrt{2}$.

Since $\operatorname{tr}^2(B) = 8 > 4$, T is hyperbolic. Its multipliers are the roots of the equation

$$z^2 + (2 - \operatorname{tr}^2(B))z + 1 = z^2 - 6z + 1 = 0,$$

hence the multipliers are

$$\{\lambda, \frac{1}{\lambda}\} = \{3 + 2\sqrt{2}, 3 - 2\sqrt{2}\}.$$

(ii) $A = \begin{pmatrix} i & 1 \\ 1 & 3i \end{pmatrix}$, $\det A = -4$, $\sqrt{\det A} = 2i$, $B = \begin{pmatrix} 1/2 & 1/(2i) \\ 1/(2i) & 3/2 \end{pmatrix}$, $\operatorname{tr} B = 2$.

Since $\operatorname{tr}^2(B) = 4$, T is parabolic. Its multipliers are the roots of the equation

$$z^2 + (2 - \operatorname{tr}^2(B))z + 1 = z^2 - 2z + 1 = 0,$$

hence the multipliers are

$$\{\lambda, \frac{1}{\lambda}\} = \{1, 1\}.$$

(iii) $A = \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}$, $\det A = i$, $\sqrt{\det A} = 1/\sqrt{2} + i/\sqrt{2}$, $B = \begin{pmatrix} 1/\sqrt{2} + i/\sqrt{2} & 1/\sqrt{2} - i/\sqrt{2} \\ 0 & 1/\sqrt{2} - i/\sqrt{2} \end{pmatrix}$, $\operatorname{tr} B = \sqrt{2}$.

Since $0 \le \operatorname{tr}^2(B) = 2 < 4$, T is elliptic. Its multipliers are the roots of the equation

$$z^2 + (2 - \operatorname{tr}^2(B))z + 1 = z^2 + 1 = 0,$$

hence the multipliers are

$$\{\lambda, \frac{1}{\lambda}\} = \{i, -i\}.$$

(iv) $A = \begin{pmatrix} -1 & 0 \\ 1 & 4 \end{pmatrix}$, $\det A = -4$, $\sqrt{\det A} = 2i$, $B = \begin{pmatrix} -1/(2i) & 0 \\ 1/(2i) & 4/(2i) \end{pmatrix}$, $\operatorname{tr} B = 3/(2i)$.

Since $\operatorname{tr}^2(B) = -9/4 < 0$, T is loxodromic. Its multipliers are the roots of the equation

$$z^2 + (2 - \operatorname{tr}^2(B))z + 1 = z^2 + \frac{17}{4}z + 1 = 0,$$

hence the multipliers are

$$\{\lambda, \frac{1}{\lambda}\} = \{-4, -\frac{1}{4}\}.$$

NAME: STUDENT NO:

Q-2) Let C be a circle in Σ passing through z_1, z_2, z_3. Prove that $I_C(z) = w$ if and only if

$$(w, z_1; z_2, z_3) = \overline{(z, z_1; z_2, z_3)}.$$

[page 55, Exercise 2M]

Solution:

Let p be the center and r the radius of C. We then have the following sequence of equations.

$$\begin{aligned}
(w, z_1; z_2, z_3) &\overset{(1)}{=} (p + \frac{r^2}{\bar{z} - \bar{p}}, z_1; z_2, z_3) \\
&\overset{(2)}{=} (\frac{r^2}{\bar{z} - \bar{p}}, z_1 - p; z_2 - p, z_3 - p) \\
&\overset{(3)}{=} (\bar{z} - \bar{p}, \frac{r^2}{z_1 - p}; \frac{r^2}{z_2 - p}, \frac{r^2}{z_3 - p}) \\
&\overset{(4)}{=} (\bar{z} - \bar{p}, \overline{z_1 - p}; \overline{z_2 - p}, \overline{z_3 - p}) \\
&= \overline{(z - p, z_1 - p; z_2 - p, z_3 - p)} \\
&\overset{(5)}{=} \overline{(z, z_1; z_2, z_3)},
\end{aligned}$$

where the equalities are justified as follows:
(1) This can be written if and only if $I_C(z) = w$ by definition of I_C.
(2) Apply the Mobius transformation $T(z) = z - p$.
(3) Apply the Mobius transformation $T(z) = r^2/z$.
(4) Each point z_i on the circle C satisfies $|z_i - p|^2 = (\overline{z_i - p})(z_i - p) = r^2$.
(5) Apply the Mobius transformation $T(z) = z + p$.

NAME: STUDENT NO:

Q-3) Show that $PSL(2,\mathbb{C})$ is generated by parabolic elements.
[page 55, Exercise 2N]

Solution:

Choose any non-parabolic element $T \in PSL(2,\mathbb{C})$. Then there exist a $\lambda \in \mathbb{C} \setminus \{0,1\}$ and $A \in PSL(2,\mathbb{C})$ such that

$$T = AU_\lambda A^{-1},$$

where

$$U_\lambda = \pm \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}.$$

Let $t = \sqrt{\lambda}$. If T is non-parabolic, then $\lambda \neq 1$, so $t \neq \pm 1$.

Now let

$$P_1 = \begin{pmatrix} p & q \\ -q & 2-p \end{pmatrix}, \text{ where } p = \frac{2}{1+t}, \; q = \pm \frac{t-1}{t+1},$$

and define

$$P_2 = U_\lambda P_1.$$

Note that P_1 and P_2 are parabolic elements of $PSL(2,\mathbb{C})$ and we have $U_\lambda = P_2 P_1^{-1}$. It now follows that

$$T = AU_\lambda A^{-1} = AP_2P_1^{-1}A^{-1} = (AP_2A^{-1})(AP_1^{-1}A^{-1}).$$

Since AP_1A^{-1} and AP_2A^{-1} are clearly parabolic elements of $PSL(2,\mathbb{C})$, we showed that every element of $PSL(2,\mathbb{C})$ is a product of two parabolic elements. This shows that $PSL(2,\mathbb{C})$ is generated by parabolic elements, as claimed.

NAME: STUDENT NO:

Q-4) Let F_θ be a rotation of S^2 by an angle θ around x_1-axis. This induces a Mobius transformation on Σ. Write this transformation explicitly. Can you generalize this by replacing x_1-axis by some other diagonal of S^2?
[Oğuz Gezmiş]

Solution:

Let f_θ be the Mobius transformation induced by F_θ. Let $\pi : S^2 \to \mathbb{C}$ be the stereographic projection from North pole,

$$\pi(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right).$$

If $\theta = \pm\pi$, then $f_\theta(0) = \infty$ and $f_\theta(i) = -i$, forcing $f_\theta(z) = 1/z$.

Now let $-\pi < \theta < \pi$. We know that f_θ is of the form

$$f_\theta(z) = \frac{az + b}{-\bar{b}z + \bar{a}}, \ |a|^2 + |b|^2 = 1.$$

From the above observation we know that for this case $a \neq 0$.

Note that

$$F_\theta(0, 0, -1) = (0, \sin\theta, -\cos\theta) \text{ and } F_\theta(0, -\sin\theta, -\cos\theta) = (0, 0, -1).$$

Since

$$\pi(0, 0, -1) = (0, 0), \pi(0, \sin\theta, -\cos\theta) = (0, \tan\frac{\theta}{2}) \text{ and } \pi(0, -\sin\theta, -\cos\theta) = (0, -\tan\frac{\theta}{2}),$$

we must have

$$f_\theta(0) = i\tan\frac{\theta}{2} \text{ and } f_\theta(-i\tan\frac{\theta}{2}) = 0.$$

Using these to solve for $a, b \in \mathbb{C}$, we find that

$$f_\theta(z) = \frac{(\cos\frac{\theta}{2})z + i\sin\frac{\theta}{2}}{(i\sin\frac{\theta}{2})z + \cos\frac{\theta}{2}}.$$