Due Date: 25 March 2013, Monday NAME:..

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Math 504 Complex Analysis II – Take-Home Exam 04 – Solutions

1	2	3	4	5	TOTAL
25	25	25	25	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.

NAME: STUDENT NO:

Q-1) Let b_1 b_2 be two complex numbers which are not congruent modΩ. Write down a function which is elliptic with respect to Ω and has poles at b_1 b_2 with principal parts

$$\frac{1}{z-b_1} + \frac{2}{(z-b_1)^2} \quad \text{and} \quad \frac{-1}{z-b_2}$$

respectively.
[page 122, Exercise 3S]

Solution:

We use Theorem 3.14.4 on page 105. Accordingly the following function is the required elliptic function:

$$f(z) = \zeta(z-b_1) + 2\mathfrak{p}(z-b_1) - \zeta(z-b_2),$$

where

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\in\Omega}{}' \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right), \text{ *(see page 94)*}$$

and

$$\mathfrak{p}(z) = -\zeta'(z) = \frac{1}{z^2} + \sum_{\omega\in\Omega}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \text{ *(see page 94)*}$$

Here $\mathfrak{p}(z)$ is the Weierstrass $\mathfrak{p}$-function which is elliptic. On the other hand $\zeta(z)$ is not elliptic but $\sum c_i\zeta(z-b_i)$ is elliptic if and only if $\sum c_i = 0$, see Lemma 3.14.3 on page 105.

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Q-2) Prove that the series $\sum_{n=1}^{\infty} z^{2^n}$ has the unit circle as its natural boundary.

[page 214, Exercise 4A]

Solution:

By ratio test we see that the series converges for $|z| < 1$.

Let α be a 2^k-th root of unity. As z approaches to α within the unit disc, all terms of the form z^{2^n} for $n \geq k$ will approach to 1 since $z^{2^n} = (z^{2^k})^{2^{n-k}}$. Hence the series will diverge at α. Since the set of all such α is dense on the unit circle, the series cannot be extended beyond $|z| = 1$ which is then the natural boundary.

There is a straightforward criterion for natural boundary in Bak and Newman, *Complex Analysis*, Springer 1999, page 231. It says that the circle of convergence of the power series $\sum_{k=0}^{\infty} c_k z^{n_k}$ is a natural boundary if $\liminf \dfrac{n_{k+1}}{n_k} > 1$. In our case $\dfrac{n_{k+1}}{n_k} = 2^k$ and we immediately conclude that $|z| = 1$ is a natural boundary.

NAME: STUDENT NO:

Q-3) Prove that the point $z = 1$ is a singular point for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

[page 214, Exercise 4B]

Solution: The shortest way to prove this is to quote Pringsham's theorem which says that $\sum_{n=1}^{\infty} a_n z^n$ has $z = R$ as a singular point if the radius of convergence is R.

On the other hand you can directly apply the short proof to this particular case. See Theorem 18.3 on page 229 of Bak and Newman quoted in question 2.

Finally on the same page there is a 3 line proof of this result as example (ii).

NAME: STUDENT NO:

Q-4) Construct the Riemann surface of $\sin^{-1} z$.

[page 214, Exercise 4C]

Solution:

We first observe the mapping property of $\sin z$.

Let $w = u + iv = \sin z = \sin x \cosh y + i \cos x \sinh y$.

We define some regions in the z-plane.

Let $\alpha < \beta$ be real numbers and let s be either 1 or -1. Then define

$$R(\alpha, \beta, s) = \{x + iy |\ \alpha \leq x \leq \beta,\ sy \geq 0\}.$$

Note that $R(\alpha, \alpha, s)$ denotes the ray $x = \alpha$ with $sy \geq 0$.

We observe that $\sin z$ sends $R(-\pi/2, \pi/2, 1)$ onto $v \geq 0$, and $R(-\pi/2, \pi/2, -1)$ onto $v \leq 0$. The rays $R(-\pi/2, -\pi/2, \pm 1)$ are both mapped onto the ray $v = 0, u \leq -1$. And the rays $R(\pi/2, \pi/2, \pm 1)$ are both mapped onto the ray $v = 0, u \geq 1$.

Moreover we observe that in the *next strip*, $\sin z$ sends $R(\pi/2, 3\pi/2, 1)$ onto $v \leq 0$, and $R(\pi/2, 3\pi/2, -1)$ onto $v \geq 0$, with the same behavior on the vertical sides.

This suggests that the Riemann surface for $\arcsin w$ is to be constructed as follows.

Let E_n be a copy of the w-plane with the rays $v = 0, u \leq -1$ and $v = 0, u \geq 1$ removed. Let

$$E = \bigcup_n E_n,$$

where lower end of the cut $v = 0, u \geq 1$ on E_n is glued to the upper end of the same cut on E_{n+1}, and the upper end of the cut $v = 0, u \leq -1$ on E_n is glued to the upper end of the same cut on E_{n-1}, for all $n \in \mathbb{Z}$. Note that $w = \pm 1$ are branch points of logarithmic type, i.e. of infinite order, since

$$\arcsin w = \frac{1}{i} \log(iw + \sqrt{1 - w^2}).$$

Finally, if $w \in E_n$, then $\arcsin w = z$ where $z \in \mathbb{C}$ is the unique point in $R(-\pi/2 + n\pi, \pi/2 + n\pi, \pm 1)$ for which $\sin z = w$.