Due Date: 17 April 2013, Wednesday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 504 Complex Analysis II – Take-Home Exam 05 – Solutions

1	2	3	4	5	TOTAL
25	25	25	25	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.

NAME: STUDENT NO:

Q-1) Find the genus of the Riemann surface of the algebraic function $w^8 + z^8 - 1 = 0$.
[page 215, Exercise 4M (a)]

Solution:

We use Theorem 4.16.3 on page 196: Let $A(z, w) = 0$ be an irreducible algebraic equation of degree n in w. Let the branch points have orders $n_1, \ldots, n_r$. Then the genus of the genus g of the Riemann surface associated to $A(z, w) = 0$ is given as

$$g = 1 - n + \frac{1}{2}\sum_{i=1}^{r} n_i.$$

Here $n = 8$. All 8^{th}-roots of unity contribute a branch of order 7 each. Since the number of branches in the plane is even, infinity does not show up as a branch. Thus

$$g = 1 - 8 + \frac{1}{2}\sum_{i=1}^{8} 7 = 21.$$

NAME: STUDENT NO:

Q-2) Find the genus of the Riemann surface of the algebraic function $w^2 - z^4(z-1) = 0$.
[page 215, Exercise 4M (b)]

Solution:

Here $n = 2$, and the branch points are 1 and -1 of order 1 each.

$$g = 1 - 2 + \frac{1}{2}(1+1) = 0.$$

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Q-3) Find the genus of the Riemann surface of the algebraic function $w^3 - w + z = 0$.
[page 215, Exercise 4M (c)]

Solution:

Here $n = 3$. To find the branch points we first find the critical points. For this we find those z for which both

$$A(z, w) = 0 \text{ and } \frac{\partial}{\partial w} A(z, w) = 0.$$

We find that critical points occur when $w = \pm 1/\sqrt{3}$ and the corresponding values of z are $\pm 2\sqrt{3}/9$. Since $\frac{\partial^2}{\partial w^2} A(z, w)$ does not vanish at these points, both of these critical points have index 1.

To see what happens at infinity we set $w = 1/u$ and $z = 1/v$, to find the equation

$$v - vu^2 + u^3 = 0.$$

Here $v = 0$ gives $u^3 = 0$, so infinity is a branch point of order 2.

Finally we can calculate the genus as

$$g = 1 - 3 + \frac{1}{2}(1 + 1 + 2) = 0.$$

NAME: STUDENT NO:

Q-4) Prove that the (real) projective plane contains a subset homeomorphic to the Mobiüs band and deduce that the projective plane is non-orientable.

[page 215, Exercise 4P]

Solution:

The real projective can be realized as the unit sphere in $\mathbb{R}^3$ with antipodal lines identified. We can write

$$\mathbb{P}^2 = S^2/\sim,$$

where S^2 is the unit sphere centered at the origin, and $p\tilde{q}$ if and only if $p = -q$.

Consider the band

$$B = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1, -\frac{1}{2} \leq y \leq \frac{1}{2} \}.$$

Clearly $B/\sim$ is a subset of $\mathbb{P}^2$ and it is a Mobius band.