Due Date: 29 April 2013, Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 504 Complex Analysis II – Take-Home Exam 06 – Solutions

1	2	3	4	5	TOTAL
25	25	25	25	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.

NAME: STUDENT NO:

Q-1) If $z = \dfrac{ai+b}{ci+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, prove that

$$\cosh \rho(i, z) = \frac{1}{2}(a^2 + b^2 + c^2 + d^2).$$

[page 268, Exercise 5F]

Solution:

We first show that $2abcd + 1 = a^2d^2 + b^2c^2$.

$$\begin{aligned} 2abcd + 1 &= 2bc(bc+1) + (ad - bc) \\ &= 2b^2c^2 + ad + bc \\ &= b^2c^2 + bc(bc+1) + ad \\ &= b^2c^2 + (ad-1)(ad) + ad \\ &= b^2c^2 + a^2d^2. \end{aligned}$$

Next we use the formula

$$\sinh^2 \frac{1}{2}\rho(z, w) = \frac{|z-w|^2}{4\,\mathrm{Im}(z)\,\mathrm{Im}(w)},$$

from page 227. We also recall the half angle formula for hyperbolic functions

$$\cosh X = 1 + 2\sinh^2 \frac{1}{2}X.$$

Putting these together we have

$$\cosh \rho(z, w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.$$

Here $w = i$, hence we have

$$\mathrm{Im}(i) = 1, \quad \mathrm{Im}(z) = \frac{1}{c^2+d^2},$$

and

$$|z-i|^2 = \left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{1-c^2-d^2}{c^2+d^2}\right)^2.$$

Putting these together we have

$$\begin{aligned} \cosh \rho(z, i) &= 1 + \frac{1}{2}\left(\frac{\left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{1-c^2-d^2}{c^2+d^2}\right)^2}{\frac{1}{c^2+d^2}}\right) \\ &= \frac{1}{2}\frac{1}{c^2+d^2}(a^2c^2 + c^4 + d^4 + 2c^2d^2 + b^2d^2 + (2abcd+1)) \\ &= \frac{1}{2}\frac{1}{c^2+d^2}(a^2c^2 + c^4 + d^4 + 2c^2d^2 + b^2d^2 + (a^2d^2 + b^2c^2)) \\ &= \frac{1}{2}(a^2 + b^2 + c^2 + d^2). \end{aligned}$$

NAME: STUDENT NO:

Q-2) Prove that there is no Fuchsian group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Can you generalize this?
[page 240, Note on top of the page.]

Solution:

Theorem 5.7.4 (page 239) says that every Abelian Fuchsian group is cyclic. Therefore no Fuchsian group can be isomorphic to a non-cyclic Abelian group such as $\mathbb{Z} \times \mathbb{Z}$.

NAME: STUDENT NO:

Q-3) Show that $PSL(2,\mathbb{R})$ is a simple group.
[page 268, Exercise 5K (iii)]

Solution:

We assume the fact that every element in $PSL(2,\mathbb{R})$ is generated by parabolic elements. For the rest first show that any normal subgroup of $SL(2,\mathbb{R})$ should stabilize all one dimensional subspaces of the real plane.

NAME: STUDENT NO:

Q-4) Show that as a topological space $PSL(2,\mathbb{R})$ is homeomorphic to $\mathbb{R}^2 \times S^1$, where S^2 is a circle.

[page 269, Exercise 5N]

Solution:

Each element of $PSL(2,\mathbb{R})$ can be written as TR where R is an elliptic element fixing i, and $T(z) = az+b$ where $a, b \in \mathbb{R}$ and $a > 0$. See pages 219 and 220 to conclude that to each such R is associated a point $e^{i\theta}$ on the unit circle. Also to T we associate the point $(\ln a, b)$ in $\mathbb{R}^2$. This gives the required isomorphism.