Due Date: 22 March, Tuesday 2016
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:....................................................

STUDENT NO:.......................................

**Math 591 Topics in Algebraic Geometry I – Homework 1 – Solutions**

| 1 | 2 | 3 | 4 | 5 | TOTAL |
|---|---|---|---|---|---|
| | | | | | |
| 50 | 50 | 0 | 0 | 0 | 100 |

*Please do not write anything inside the above boxes!*

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
**Submit your solutions on this booklet only. Use extra pages if necessary.**

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**Rules for Homework Assignments**

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you write your answers alone.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

**Affidavit of compliance with the above rules:** I affirm that I have complied with the above rules in preparing this submitted work.

*Please sign here:*

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**Q-1)** A plane curve is defined to be an irreducible algebraic set of dimension one in $\mathbb{P}^2$. Show that any two plane curves intersect.

**Solution:**

We prove something slightly more general. Let $X$ and $Y$ be two projective hypersurfaces in $\mathbb{P}^n$. We want to show that $X$ and $Y$ always intersect non-trivially.

We recall from Exercise 3.5 on page 21 that $\mathbb{P}^n \setminus X$ is affine. If $Y$ does not intersect $X$ then $Y$ lies in $\mathbb{P}^n \setminus X$ and is affine. The only variety which is both projective and affine is a point, see Exercise 3.1(e) on page 21. But when $n \geq 2$, the dimension of $Y$ is at least 1, and hence $Y$ cannot be a point. This contradiction shows that $X$ and $Y$ must intersect non-trivially when $n \geq 2$.

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**Q-2)** Let $X \subset \mathbb{A}^n$ be a variety of dimension $r$. Consider the two sets of generators for the ideal of $X$,

$$I(X) = (f_1, \ldots, f_s) = (g_1, \ldots, g_t),$$

where each $f_i$ and $g_j$ are polynomials in $k[x_1, \ldots, x_n]$. Choose a point $p \in X$ and define the Jacobian matrices

$$J_f = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}}, \quad \text{and} \quad J_g = \left(\frac{\partial g_i}{\partial x_j}(p)\right)_{\substack{1 \leq i \leq t \\ 1 \leq j \leq n}}.$$

Show that $\operatorname{rank} J_f = \operatorname{rank} J_g$.

**Solution:**

Since each $g_i$ is in $(f_1, \ldots, f_s)$, there exist polynomials $h_{i1}, \ldots, h_{is}$ such that

$$g_i = h_{i1} f_1 + \cdots + h_{is} f_s,$$

and for any $j = 1, \ldots, n$,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial h_{i1}}{\partial x_j} f_1 + h_{i1} \frac{\partial f_1}{\partial x_j} + \cdots + \frac{\partial h_{is}}{\partial x_j} f_s + h_{is} \frac{\partial f_s}{\partial x_j}.$$

Evaluating this expression at a point $p \in X$ and noting that each $f_i$ vanish on $X$, we get

$$\frac{\partial g_i}{\partial x_j}(p) = h_{i1}(p) \frac{\partial f_1}{\partial x_j}(p) + \cdots + h_{is}(p) \frac{\partial f_s}{\partial x_j}(p).$$

This shows that each row of $J_g$ is a linear combination of rows of $J_f$. Hence the row space of $J_g$ is a vector subspace of the row space of $J_f$. This means that $\operatorname{rank} J_g \leq \operatorname{rank} J_f$. From symmetry we get the other inequaly also. Hence the equality of the ranks follows.