Solutions to written works 2, 3 and Final Take-Home Exam

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The following exercises are from Hartshorne's *Algebraic Geometry*

I.1.6

Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\bar{Y}$ is also irreducible.

Let X be an irreducible topological space and Y a nonempty open subset. If $Y = X$ then Y is irreducible and dense as claimed. If however $Y \subsetneq X$, then $X \setminus Y$ is a nonempty closed proper subset of X and we can write $X = \overline{Y} \cup (X \setminus Y)$. Since X is irreducible we must have $X = \overline{Y}$, i.e. Y is dense in X. Assume $Y = Y_1 \cup Y_2$ where Y_i are closed subsets of Y. There are then closed subsets X_1, X_2 of X such that $Y_i = Y \cap X_i$. If $Y_i \subsetneq Y$, then $X_i \subsetneq X$ and $X = X_1 \cup X_2$ renders X reducible which is not the case. Hence either Y_1 or Y_2 must be equal to Y and Y is irreducible. (This proves the first part of the problem which is Example 1.1.3.)

Next assume that $\overline{Y}$ is not irreducible when Y is irreducible in its induced topology. Then $\overline{Y} = Y_1 \cup Y_2$ where Y_i are proper closed subsets of $\overline{Y}$, and we have $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$. Since Y is irreducible one of these subsets is not proper, say $Y = Y \cap Y_1$. Then $Y \subset Y_1$ and $\overline{Y} \subset Y_1$ since Y_1 is closed but this contradicts the choice of Y_1 as a proper subset of $\overline{Y}$. Hence $\overline{Y}$ is irreducible. (This second part of the problem is Example 1.1.4.)

I.1.8

Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$. (See (7.1) for a generalization.)

Let Y be the zero set of the prime ideal $\mathfrak{p} \subset k[x_1, \ldots, x_n]$. Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial such that $H = Z(f)$. Set $B = k[x_1, \ldots, x_n]/\mathfrak{p}$. Let $\bar{f}$ denote the residue of f in B. Then $\bar{f} = \bar{f}_1 \cdots \bar{f}_s$ where each f_i is a polynomial in $k[x_1, \ldots, x_n]$ and $\bar{f}_i$ is irreducible in B. We have the decomposition $(f) + \mathfrak{p} = [(f_1) + \mathfrak{p}] \cdots [(f_s) + \mathfrak{p}]$ and each irreducible component Y_i of $Y \cap H$ is of the form $Z((f_i) + \mathfrak{p})$. The coordinate ring of Y_i is $k[x_1, \ldots, x_n]/[(f) + \mathfrak{p}]$ and is isomorphic to $B/(\bar{f}_i)$. None of the $\bar{f}_i$ is a unit or a zero divisor in B, and being irreducible generates a prime ideal. Hence $\text{height}(\bar{f}_i) = 1$ in B, by Theorem I.1.11A. Now invoking Theorem I.1.8A we have $\text{height}(\bar{f}_i) + \dim B/(\bar{f}_i) = \dim B$. It follows that $\dim B/(\bar{f}_i) = r - 1$ which is the dimension of the irreducible component Y_i of $Y \cap H$, $i = 1 \ldots, s$.

I.1.9

Let $\mathfrak{a} \subseteq A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

We will prove this by induction. When $r = 1$, $\mathfrak{a}$ is generated by a single polynomial $f = g_1 \cdots g_s$ where each g_i is irreducible in A. An irreducible component of $Z(\mathfrak{a})$ is of the form $Z(g_i)$ and is of dimension $n - 1$ by Proposition I.1.13.

Now assume that the statement is true for $r - 1$. Let $\mathfrak{a}$ be generated by $f_1, \ldots, f_{r-1}$, and f. An irreducible component X of $Z(\mathfrak{a})$ is of the form $Y \cap Z(g)$ where Y is an irreducible component of $Z(f_1, \ldots, f_{r-1})$ and g is an irreducible factor of f. Then by the previous exercise (I.1.8), $\dim X = \dim Y - 1$. Moreover $\dim Y \geq n - (r-1)$ by the induction hypothesis. Putting these together, we find that $\dim X \geq n - r$ as claimed. This completes the proof by induction.

I.1.11

Let $Y \subseteq \mathbb{A}^3$ be a curve given parametrically by $x = t^3$, $y = t^4$, $z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection* –cf. (Ex. 2.17).

Since $t \mapsto (t^3, t^4, t^5)$ is a homeomorphism of $\mathbb{A}^1$ with Y, we have that $\dim Y = 1$. Let $B = k[x, y, z]$ and $\mathfrak{p} = I(Y)$. Then $\dim Y = \dim B/\mathfrak{p} = 1$. From the identity (see Theorem 1.8A) $\text{height}\, \mathfrak{p} + \dim B/\mathfrak{p} = \dim B$, we get $\text{height}\, \mathfrak{p} = 2$, since clearly $\dim B = 3$. By trial and error we find that the polynomials of smallest degree in $\mathfrak{p}$ are all linear combinations of $xz - y^2$, $yz - x^3$ and $z^2 - x^2 y$. Since these three polynomials are linearly independent, no two of them can generate $\mathfrak{p}$. Hence $I(Y)$ cannot be generated by two elements even though its height is two.

I.1.12

Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$, whose zero set $Z(f)$ in $\mathbb{A}^2_{\mathbb{R}}$ is not irreducible (cf. 1.4.2).

The simplest case is when $f = x^2 + y^2 + 1$ whose zero set is empty in $\mathbb{A}^2_{\mathbb{R}}$ and empty set is not considered irreducible.

A more convincing example can be constructed as follows. Take two real numbers a and b, not both zero. Then consider the polynomial $f = (x^2 - a^2)^2 + (y^2 - b^2)^2 = [(x^2 - a^2) + i(y^2 - b^2)][(x^2 - a^2) - i(y^2 - b^2)]$. It is straightforward to show that these factors are irreducible in $\mathbb{C}[x, y]$. Since the factorization in $\mathbb{C}[x, y]$ is unique, f cannot be factored into real components and is thus f irreducible in $\mathbb{R}[x, y]$. The zero set of f now has two components if $ab = 0$ and has four components otherwise.

I.2.12

***The d-Uple Embedding.* For given $n, d > 0$, let $M_0, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the *d-uple embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1$, $d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.**

(a) Let $\theta : k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

(c) Now show that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

(a)
Being the kernel of a homomorphism, $\mathfrak{a}$ is prime. We need to show that it is homogeneous. Let $f = f_0 + \cdots + f_m \in \mathfrak{a}$, where each f_i is homogeneous of degree i in the variables $y_0, \ldots, y_N$, and $m = \deg f$. We have $f(M_0, \ldots, M_N) = f_0(M_0, \ldots, M_N) + \cdots + f_m(M_0, \ldots, M_N) = 0$ identically. But each $f_i(M_0, \ldots, M_N)$ is a homogeneous polynomial of degree id in the variables $x_0, \ldots, x_n$. Since they all have different degrees, they cannot cancel each other and hence they must be all identically zero. This gives $f_i \in \mathfrak{a}$, for

$i = 0, \dots, m$, showing that $\mathfrak{a}$ is a homogeneous ideal.

(b)
For this problem we fix an ordering of the monomials of degree d in the variables $x_0, \dots, x_n$. The map ρ_d is defined with this ordering. In particular if

$$\rho_d(x) = \rho_d([x_0 : \cdots : x_n]) = [\cdots : x_{i_1}^{j_1} \cdots x_{i_r}^{j_r} : \cdots] = y,$$

then the homogeneous coordinate of y corresponding to the place of $x_{i_1}^{j_1} \cdots x_{i_r}^{j_r}$ will be denoted by $\left\{ y \middle| \begin{matrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{matrix} \right\}$. Here of course the $i_1, \dots, i_r, j_1, \dots, j_r$ are integers with $0 \le i_1 < \cdots < i_r \le n$, and $j_1, \dots, j_r \ge 0$ with $j_1 + \cdots + j_r = d$. We will refer to such a collection of indices as admissible.

We want to show that $\rho_d(\mathbb{P}^n) = Z(\mathfrak{a})$.

If $y = \rho_d(x)$ for some $x \in \mathbb{P}^n$, then the entries of y are monomials of degree d in the variables $x_0, \dots, x_n$, hence any polynomials $f \in \mathfrak{a}$ which vanishes on these monomials will vanish at y. This shows that $y \in Z(\mathfrak{a})$, and the inclusion $\rho_d(\mathbb{P}^n) \subset Z(\mathfrak{a})$ is easily established.

Now assume that we have a point $a = [\cdots : \left\{ a \middle| \begin{matrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{matrix} \right\} : \cdots] \in Z(\mathfrak{a})$. We will show that there exists a point $c = [c_0 : \cdots : c_n] \in \mathbb{P}^n$ such that

$$\left\{ a \middle| \begin{matrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{matrix} \right\} = c_{i_1}^{j_1} \cdots c_{i_r}^{j_r},$$

for all admissible indices.

First observe that for any collection of admissible indices, we have

$$\left\{ y \middle| \begin{matrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{matrix} \right\}^d - \left\{ y \middle| \begin{matrix} d \\ i_1 \end{matrix} \right\}^{j_1} \cdots \left\{ y \middle| \begin{matrix} d \\ i_r \end{matrix} \right\}^{j_r} \in \mathfrak{a}.$$

Since $a \in Z(\mathfrak{a})$, we must have

$$\left\{ a \middle| \begin{matrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{matrix} \right\}^d = \left\{ a \middle| \begin{matrix} d \\ i_1 \end{matrix} \right\}^{j_1} \cdots \left\{ a \middle| \begin{matrix} d \\ i_r \end{matrix} \right\}^{j_r},$$

so not all of

$$\left\{ a \middle| \begin{matrix} d \\ 0 \end{matrix} \right\}, \dots, \left\{ a \middle| \begin{matrix} d \\ n \end{matrix} \right\}$$

can be zero. After reordering the monomials if necessary and multiplying by a suitable caontant we can assume that

$$\left\{ a \,\middle|\, \begin{matrix} d \\ 0 \end{matrix} \right\} = 1.$$

We now pick a point $c = [c_0 : \cdots : c_n] \in \mathbb{P}^n$ as follows.

$$c_0 = 1, \quad c_i = \left\{ a \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i \end{matrix} \right\}, \text{ for } i = 1, \ldots, n.$$

We will show that

$$a = \rho_d(c),$$

which will show that $Z(\mathfrak{a}) \subset \rho_d(\mathbb{P}^n)$ and will finish the proof.

We will show that

$$\left\{ a \,\middle|\, \begin{matrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{matrix} \right\} = c_{i_1}^{j_1} \cdots c_{i_r}^{j_r},$$

keeping in mind that $c_0 = 1$.

First observe that since

$$\left\{ y \,\middle|\, \begin{matrix} d-j & j \\ 0 & i \end{matrix} \right\} \left\{ y \,\middle|\, \begin{matrix} d \\ 0 \end{matrix} \right\}^{j-1} - \left\{ y \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i \end{matrix} \right\}^{j} \in \mathfrak{a},$$

we must have

$$\left\{ a \,\middle|\, \begin{matrix} d-j & j \\ 0 & i \end{matrix} \right\} = \left\{ a \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i \end{matrix} \right\}^{j} = c_i^j.$$

Moreover, in general we have

$$\left\{ y \,\middle|\, \begin{matrix} j_1 & j_2 & \cdots & j_r \\ 0 & i_2 & \cdots & i_r \end{matrix} \right\} \left\{ y \,\middle|\, \begin{matrix} d \\ 0 \end{matrix} \right\}^{r-j_1-1} - \left\{ y \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i_2 \end{matrix} \right\}^{j_2} \cdots \left\{ y \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i_r \end{matrix} \right\}^{j_r} \in \mathfrak{a},$$

and

$$\left\{ y \,\middle|\, \begin{matrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{matrix} \right\} \left\{ y \,\middle|\, \begin{matrix} d \\ 0 \end{matrix} \right\}^{r-1} - \left\{ y \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i_1 \end{matrix} \right\}^{j_1} \cdots \left\{ y \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i_r \end{matrix} \right\}^{j_r} \in \mathfrak{a}.$$

It then follows that

$$\begin{aligned} \left\{ a \,\middle|\, \begin{matrix} j_1 & j_2 & \cdots & j_r \\ 0 & i_2 & \cdots & i_r \end{matrix} \right\} \left\{ a \,\middle|\, \begin{matrix} d \\ 0 \end{matrix} \right\}^{r-j_1-1} &= \left\{ a \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i_2 \end{matrix} \right\}^{j_2} \cdots \left\{ a \,\middle|\, \begin{matrix} d-1 & 1 \\ 0 & i_r \end{matrix} \right\}^{j_r} \\ &= c_{i_2}^{j_2} \cdots c_{i_r}^{j_r}, \end{aligned}$$

and

$$\begin{aligned}\left\{a\left|\begin{matrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r\end{matrix}\right.\right\}\left\{a\left|\begin{matrix} d \\ 0\end{matrix}\right.\right\}^{r-1} &= \left\{a\left|\begin{matrix} d-1 & 1 \\ 0 & i_1\end{matrix}\right.\right\}^{j_1}\cdots\left\{a\left|\begin{matrix} d-1 & 1 \\ 0 & i_r\end{matrix}\right.\right\}^{j_r} \\ &= c_{i_1}^{j_1}\cdots c_{i_r}^{j_r}.\end{aligned}$$

This completes the proof.

(c)
We first show that ρ_d is one-to-one. Let $c = [c_0 : \cdots : c_n]$ and $e = [e_0 : \cdots : e_n]$ are points of $\mathbb{P}^n$ such that $\rho_d(c) = \rho_d(e)$. At least one of the homogeneous coordinates e_i is different than zero. For convenience of notation assume $e_0 \neq 0$. Since the images of c and e agree under ρ_d, we must have $c_0^d = e_0^d$ and $c_0^{d-1}c_i = e_0^{d-1}e_i$ for $i = 1, \ldots, n$. From $c_0^d = e_0^d$, we get $c_0 = \omega e_0$, where $\omega^d = 1$. From the other equations we get $c_i = \omega e_i$ for $i = 1, \ldots, n$. This shows that $c = e$ and hence ρ_d is one-to-one.

Next we show that ρ_d is continuous. In fact if $Z(f_1, \ldots, f_m)$ is a closed subset of $Z(\mathfrak{a})$, where $f_1, \ldots, f_m$ are homogeneous polynomials in $y_0, \ldots, y_N$, then $\rho_d^{-1}(Z(f_1, \ldots, f_m)) = Z(f_1 \circ \rho_d, \ldots, f_m \circ \rho_d)$ is closed. Thus ρ_d is continuous.

We now have a continuous isomorphism from $\rho_d(\mathbb{P}^n)$ onto $Z(\mathfrak{a})$. It is well known that a continuous isomorphism is not necessarily a homeomorphism. Therefore we have to check separately if ρ_d is a homeomorphism in this case.

What remains to be shown for ρ_d to be a homeomorphism is that it maps closed sets to closed sets. Since every ideal in $k[x_0, \ldots, x_n]$ is finitely generated, every closed set is an intersection of finitely many hypersurfaces. It therefore suffices to show that a hypersurface is mapped under ρ_d to a closed set in $Z(\mathfrak{a})$. And for this we need to show that for any homogeneous polynomial $g \in k[x_0, \ldots, x_n]$, there corresponds a homogeneous polynomial $G \in k[y_0, \ldots, y_N]$ such that $\rho_d(Z(g)) = Z(\mathfrak{a}) \cap Z(G)$. We now describe a way of obtaining G from g.

Let $g \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree m. Let

$$w_t = x_0^{i_{0t}} x_1^{i_{1t}} \cdots x_n^{i_{nt}},\ t = 1, \ldots, d.$$

be homogeneous monomials, not necessarily distinct, occurring in g with non-zero coefficients, where $i_{0t}, \ldots, i_{nt}$ are non-negative integers with $i_{0t} + \cdots +$

$i_{nt} = m$ for $t = 1, \ldots, d$. A typical monomial occurring in the polynomial g^d is of the form

$$w = w_1 \cdots w_d = x^{i_{01}+\cdots+i_{0d}} \cdots x^{i_{n1}+\cdots+i_{nd}}.$$

For $s = 0, \ldots, n$, let

$$i_{s1} + \cdots + i_{sd} = u_s d + v_s,$$

where $0 \le u_s$ and $0 \le v_s < d$. Then we can write

$$w = (x_0^d)^{u_0} \cdots (x_n^d)^{u_n} (x_0^{v_0} \cdots x_n^{v_n}).$$

Since the degree of the monomial w is dm, we see that $v_0 + \cdots + v_n = \ell d$, where $\ell \ge 0$ is a non-negative integer. Since each $v_s < d$, we have that $0 \le \ell \le n$. Therefore there exist integers $0 \le j_0 < \cdots < j_\ell \le n$ such that we can write

$$v_{j_e} = v'_{j_e} + v''_{j_e},\ e = 0, \ldots, \ell$$

in such a way that

$$v_0 + \cdots + v_n = (v_0 + \cdots + v_{j_0-1} + v'_{j_0}) + (v''_{j_0} + v_{j_0+1} + \cdots + v_{j_1-1} + v'_{j_1}) + \cdots + (v''_{j_\ell} + v_{j_\ell+1} + \cdots + v_n)$$

in such a way that each parenthesis adds up to d. Then

$$x_0^{v_0} \cdots x_n^{v_n} = (x_0^{v_0} \cdots x_{j_0}^{v'_{j_0}}) \cdots (x_{j_\ell}^{v''_{j_\ell}} \cdots x_n^{v_n}).$$

This proves that the monomial w of degree dm can be written as a product of monomials of degree d in the variables $x_0, \ldots, x_n$. In fact let $M_0, \ldots, M_N$ be a list of monomials of degree d in $x_0, \ldots, x_n$, and let ϕ be the map

$$\phi : k[M_0, \ldots, M_N] \to k[y_0, \ldots, y_N],$$

sending each M_i to y_i, then $\phi(w)$ is a monomial of degree m in the variables $y_0, \ldots, y_N$. Define the polynomial G as

$$G(y_0, \ldots, y_N) = \phi(g^d(x_0, \ldots, x_n) \in k[y_0, \ldots, y_N],$$

where the above process of writing g^d as a polynomial in M_i is understood, before ϕ is applied. Then G is uniquely defined and is homogeneous of degree m.

It is now clear from the description of the maps that for any point $a = [a_0 : \cdots : a_n] \in \mathbb{P}^n$,

$$G(\rho_d(a)) = \phi(g^d(a)).$$

It follows from this that $a \in Z(g)$ if and only if $\rho_d(a) \in Z(G) \cap Z(\mathfrak{a})$. This shows that the closed set $Z(g)$ is mapped onto the closed set $Z(G) \cap Z(\mathfrak{a})$, which completes the proof that ρ_d is a closed map.

Thus ρ_d is a homeomorphism from $\mathbb{P}^n$ onto $Z(\mathfrak{a})$

(d)
Let $[s : t]$ and $[x_0 : x_1 : x_2 : x_3]$ be the homogeneous coordinates in $\mathbb{P}^1$ and $\mathbb{P}^3$ respectively. Let $\rho_3([s : t]) = [s^3 : s^2t : s^2t^2 : st^3 : t^3]$, and $\phi : U \subset \mathbb{P}^1 \to \mathbb{A}^1$ be given as usual by $\phi([s : t]) = t/s$. Then $Y = \rho_3 \circ \phi^{-1}(t) = [1 : t : t^2 : t^3]$ is an embedding of $\mathbb{A}^1$ into $\mathbb{P}^2$ such that $\bar{Y}$ is the twisted cubic. Note that Y also coincides with $\rho_3(U)$. The only point in $\mathbb{P}^1$ not in U is $[0 : 1]$ which maps under ρ_3 to $[0 : 0 : 0 : 1]$. Since ρ_3 is an isomorphism onto its image, the closure of $\rho_3(U)$ contains only this extra point. Hence $\bar{Y}$ is $\rho_3(\mathbb{P}^1)$.

I.2.14

***The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_ib_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a *subvariety* of $\mathbb{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} | i = 0, \dots, r,\ j = 0, \dots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \dots, x_r, y_0, \dots, y_s]$ which sends z_{ij} to x_iy_j. Then show that $\operatorname{Im}\psi = Z(\mathfrak{a})$.]**

That ψ is well defined is immediate since if $a_i \neq 0$ and $b_j \neq 0$, then $z_{ij} = x_iy_j \neq 0$. Moreover if $\psi(a_0, \dots, a_r) \times (b_0, \dots, b_s) = (\dots, z_{ij}, \dots)$, then $\psi(\lambda a_0, \dots, \lambda a_r) \times (\mu b_0, \dots, \mu b_s) = (\dots, \lambda\mu a_ib_j, \dots)$. Hence ψ gives the same point in $\mathbb{P}^N$ independent of which representative is used for the point $a \times b \in \mathbb{P}^r \times \mathbb{P}^s$.

To show injectivity, assume $\psi(a \times b) = \psi(a' \times b')$. Then for some i, j, we have $a_i \neq 0$ and $b_j \neq 0$. Since $z_{ij} = z'_{ij}$, we have $a_ib_j = a'_ib'_j$, hence $a'_i \neq 0$ and $b'_j \neq 0$. Without loss of generality assume then that $a_i = b_j = a'_i = b'_j = 1$. We have $z_{uj} = a_u$ and $z'_{uj} = a'_u$ for $u = 0, \dots, r$. Hence, $a = a'$. Similarly, exploiting the fact that $z_{iv} = z'_{iv}$ for $v = 0, \dots, s$ gives $b = b'$. Thus ψ is injective.

Now we show that $\operatorname{Im}\psi = Z(\mathfrak{a})$.

Let $p = (\dots, p_{ij}, \dots) \in \operatorname{Im}\psi$. Since $p_{ij} = x_i y_j$ for some $x = (x_0, \dots, x_r) \in \mathbb{P}^r$ and some $y = (y_0, \dots, y_s) \in \mathbb{P}^s$, for every $f \in \mathfrak{a}$, we have $f(p) = 0$. Hence $p \in Z(\mathfrak{a})$.

Conversely, assume that $p = (\dots, p_{ij}, \dots) \in Z(\mathfrak{a})$. Without loss of generality assume that $p_{00} = 1$. Since the polynomial $z_{00}z_{uv} - z_{u0}z_{0v}$ is in $\mathfrak{a}$, we have $p_{uv} = p_{u0}p_{0v}$ for all $u = 0, \dots, r$ and $v = 0, \dots, s$. Let $a = (1, p_{10}, \dots, p_{r0}) \in \mathbb{P}^r$ and $b = (1, p_{01}, \dots, p_{0s}) \in \mathbb{P}^s$. Then $p = \psi(a \times b)$, hence $p \in \operatorname{Im}\psi$.

Thus we get $\operatorname{Im}\psi = Z(\mathfrak{a})$.

Remark: We now prove that $\mathfrak{a}$ is generated by all polynomials of the form $z_{i_1j_1}z_{i_2j_2} - z_{i_1j_2}z_{i_2j_1}$, where i_1, i_2 run through 0 to r, and j_1, j_2 run through 0 to s. Denote this ideal by J. We will show that $J = \mathfrak{a}$. Clearly $J \subseteq \mathfrak{a}$.

We want to show that $\mathfrak{a} \subseteq J$. For this, pick any homogeneous polynomial f in $\mathfrak{a}$. Let $M = z_{i_1j_1} \cdots z_{i_nj_n}$ be a homogeneous form of degree n appearing in f with some non-zero constant coefficient.

Since f vanishes on every point of $\operatorname{Im}\Psi$, there must be another homogeneous form M' appearing in f with the same coefficient multiplied by -1 such that if $M' = z_{u_1v_1} \cdots z_{u_nv_n}$, then $\{i_1, \dots, i_n\} = \{u_1, \dots, u_n\}$ and $\{j_1, \dots, j_n\} = \{v_1, \dots, v_n\}$.

We know claim that $M - M'$ is in J. We do this by induction on the degree n.

If $n = 2$, the statement is trivial.

Assume $n > 2$. Both M and M' evaluated on $\operatorname{Im}\psi$ will be equal to

$$x_{i_1} \cdots x_{i_n} y_{j_1} \cdots y_{j_n}.$$

In particular x_{u_n} and y_{v_n} are in this list, corresponding to $z_{u_nv_n}$ term of M'. So M has at least two terms of the form z_{u_nj}, z_{iv_n} for some i and j. Without loss of generality assume that they are $z_{i_{n-1}j_{n-1}}$ and $z_{i_nj_n}$. i.e. assume that $u_n = i_{n-1}$ and $v_n = j_n$.

Then we can write

$$\begin{aligned} M - M' &= (z_{i_1j_1}\cdots z_{i_{n-2}j_{n-2}})[z_{i_{n-1}j_{n-1}} z_{i_nj_n} - z_{i_nj_{n-1}} \overbrace{z_{i_{n-1}j_n}}^{z_{u_nv_n}}] \\ &\quad - \left(z_{i_1j_1}\cdots z_{i_{n-2}j_{n-2}} z_{i_nj_{n-1}} - z_{u_1v_1}\cdots z_{u_{n-1}v_{n-1}}\right) z_{u_nv_n}. \end{aligned}$$

Now check that the term inside the parenthesis on the second line is homogeneous of degree $n-1$ and satisfies the conditions about x_i and y_j as $M - M'$ satisfies. By induction hypothesis, it is inside J. It is now clear that $M - M' \in J$. This shows that $\mathfrak{a} \subseteq J$ and concludes the proof that they are equal.

I.2.16

(a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbb{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic and a line.

(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

(a)
Let $[w : x : y : z]$ be the homogeneous coordinates in $\mathbb{P}^3$.

If $x = 0$ then either $y = 0$ or $w = 0$. In the subcase $y = 0$, we get the points $[1 : 0 : 0 : 0]$ and $[0 : 0 : 0 : 1]$. In the subcase $w = 0$, we get the line $[0 : 0 : y : z]$ where $[y : z] \in \mathbb{P}^1$.

If $x \neq 0$, then $wyz \neq 0$. Without loss of generality we assume that $w = 1$ and get the solution space $[1 : x : x^2 : x^3]$ where $x \in k$, which is the twisted cubic in $\mathbb{P}^3$ together with the point $[0 : 0 : 0 : 1]$.

(b)
The intersection of C and L is easily seen to be the point $P = [0 : 0 : 1]$ if $[x : y : z]$ denotes the homogeneous coordinates of $\mathbb{P}^2$. It is also clear that $I(P) = (x, y)$. But $I(C) + I(L) = (x^2 - yz, y) = (x^2, y) \subsetneq (x, y) = I(P)$.

I.2.17

***Complete Intersections.* A variety Y of dimension r in $\mathbb{P}^n$ is *(strict) complete intersection* if $I(Y)$ can be generated by $n-r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n-r$ hypersurfaces.**

(a) Let Y be a variety in $\mathbb{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geq n-q$.

(b) Show that a strict complete intersection is a set-theoretic complete intersection.

$*$(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbb{P}^3$ (EX. 2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1 and H_2 of degrees $2, 3$ respectively, such that $Y = H_1 \cap H_2$.

$**$(d) It is an unsolved problem whether every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic intersection of two surfaces. See Hartshorne [1] and Hartshorne [5, III. §5] for commentary.

(a)
We will prove this by induction on q. When $q = 1$, this is the content of Ex. I.2.8.

Assume true for $q-1$. And let f_q be an irreducible polynomial not in $(f_1, \ldots, f_{q-1})$. Let $X = Z(f_1, \ldots, f_{q-1})$ and $Y = Z(f_1, \ldots, f_q)$, and let $S(X) = k[x_0, \ldots, x_n]/(f_1, \ldots, f_{q-1})$ and $S(Y) = k[x_0, \ldots, x_n]/(f_1, \ldots, f_q)$ be the corresponding projective coordinate rings.

From Ex. I.2.6 we know that $\dim S(X) = \dim X + 1$ and $\dim S(Y) = \dim Y + 1$. By induction hypothesis, we have $\dim S(X) = \dim X + 1 \geq [n-(q-1)]+1$. We will show that $\dim S(Y) \geq [n-q]+1$.

Let $\mathfrak{p}$ be the prime ideal in $S(X)$ generated by f_q. Note that we have

$$S(Y) = S(X)/\mathfrak{p}.$$

By Theorem I.8A.b applied to the right hand side of this equation, we have

$$\dim S(Y) = \dim S(X) - \operatorname{height} \mathfrak{p}.$$

By Theorem I.1.11A, height $\mathfrak{p} = 1$. Putting in this with the induction hypothesis that $\dim S(X) \geq [n-(q-1)]+1$, we get

$$\dim S(Y) \geq n - q + 1,$$

as required.

(b)
Let Y be a strict complete intersection in $\mathbb{P}^n$ of dimension r. Then $I(Y) = (f_1, \ldots, f_{n-r})$ where each f_i is a homogeneous polynomial in the variables $x_0, \ldots, x_n$. Let $H_i = Z(f_i)$ be the hypersurface defined by f_i, for $i = 1, \ldots, n-r$. We then have

$$Y \subset H_1 \cap \cdots \cap H_{n-r}.$$

Conversely, let p be a point of the intersection of the hypersurfaces. Then $f_i(p) = 0$ for all $i = 1, \ldots, n-r$. This says that $p \in Y$. Hence Y is the set theoretic intersection of the hypersurfaces $H_1, \ldots, H_{n-r}$.

(c)
Let $Y \in \mathbb{P}^3$ be the twisted cubic parametrized by

$$[s : t] \mapsto [s^3 : s^2t : st^2 : t^3] \in \mathbb{P}^3,$$

where $[s : t] \in \mathbb{P}^1$. Using $[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3$ as the homogeneous coordinates, set

$$f_1 = x_0x_2 - x_1^2, \quad f_2 = x_1x_3 - x_2^2, \quad f_3 = x_0x_3 - x_1x_2.$$

It is easy to show that

$$I(Y) = (f_1, f_2, f_3).$$

Moreover let

$$\phi = x_3f_3 - x_2f_2.$$

It is also straightforward to see that

$$Y = Z(f_1) \cap Z(\phi).$$

But

$$I(Y) \supsetneq I(f_1, \phi)$$

since f_2 is not an element of $I(f_1, \phi)$. (Check however that $f_2^2 = x_2\phi - x_3^2 f_1$ and so belongs to $I(f_1, \phi)$.)

(d)
This problem is still unsolved at the time of this writing, 2016.

I.3.8

Let H_i and H_j be hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbb{P}^n - (H_i \cap H_j)$ is constant. (This gives an alternate proof of (3.4a) in the case $Y = \mathbb{P}^n$.)

Let ϕ be a regular function on $\mathbb{P}^n - (H_i \cap H_j)$. Restricting ϕ to $\mathbb{P}^n - H_i$ we get a regular function there. Bur regular functions on $\mathbb{P}^n - H_i$ are of the form $\dfrac{f}{x_i^m}$ where f is a homogeneous polynomial in $x_0, \ldots, x_n$ of degree $m \geq 0$, and $x_i \not| f$. Similarly restricting ϕ to $\mathbb{P}^n - H_j$ we get a regular function of the form $\dfrac{g}{x_j^r}$ where g is homogeneous of degree $r \geq 0$ and $x_j \not| g$. On the intersection $(\mathbb{P}^n - H_i) \cap (\mathbb{P}^n - H_j)$ we must have $\dfrac{f}{x_i^m} = \dfrac{g}{x_j^r}$. This gives $f x_j^r = g x_i^m$. But since $i \neq j$, x_j does not divide the right hand side. This forces $r = 0$. Similarly $m = 0$. So $f = g$ are homogeneous of degree 0, i.e. they are constants. On the other hand, ϕ being a constant on an open set is constant throughout.

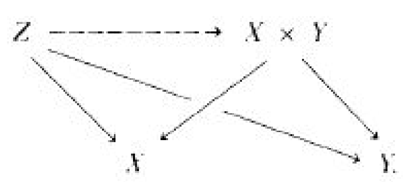
I.3.15

Products of Affine Varieties. **Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.**

(a) Show that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible. [*Hint*: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$, $i = 1, 2$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the *product* of X and Y. Note that its topology is in general not equal to the product topology (Ex. 1.4).

(a) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties, i.e., show (i) the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and (ii) given a variety Z, and morphisms $Z \to X$, $Z \to Y$, there is a unique morphism $Z \to X \times Y$ making a commutative diagram



(c) Show that $\dim X \times Y = \dim X + \dim Y$.

(a)
We follow the hint. Assume first that there is an $x_0 \in X$ such that x_0 is

neither in X_1 nor in X_2. Then define the sets $Y_i = \{y \in Y \mid x_0 \times y \in Z_i\}$, $i = 1, 2$. Now $Y = Y_1 \cup Y_2$, and we show that Y_1, Y_2 are closed subsets of Y. Fix $i = 1, 2$. Let us use the notation $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ for coordinates in $\mathbb{A}^{n+m}$. Let J_i be the ideal in $k[y]$ generated by the polynomials of the form $f_i(x_0, y)$ for all $f_i(x, y)$ belonging to the ideal of Z_i in $k[x, y]$. Then the zero set of J_i is precisely Y_i, so Y_i is closed. But Y is irreducible, so $Y = Y_1$ or Y_2, which implies that x_0 is in X_1 or in X_2, which is in contradiction to the way x_0 was chosen. So no such x_0 can be chosen and $X = X_1 \cup X_2$.

Next we show that X_1 is closed. Let J_1 be the ideal in $k[x]$ generated by polynomials $f(x, y_0)$ where $f(x, y)$ is a polynomial vanishing on Z_1 and y_o is a point on Y. Then X_1 is the zero set of J_1 and is therefore closed. Similarly X_2 is closed. But X being irreducible either $X = X_1$ or $X = X_2$. But this implies that either $X \times Y = Z_1$ or $X \times Y = Z_2$, showing that $X \times Y$ is irreducible.

(b)
Define a map $\phi : A(X) \otimes_k A(Y) \to A(X \times Y)$ as $\phi(\sum f_i(x) \otimes g_i(y)) = \sum f_i(x) g_i(y)$. This is a ring homomorphism. It is onto since $\phi(x_i \otimes y_j) = x_i y_j$ and they generate the ring $A(X \times Y)$. Now let r be the smallest integer such that there exist $F = \sum_{i=1}^{r} f_i(x) \otimes g_i(y)$ with $\phi(F) = 0$. From the minimality of r, we see that the g_i are not in the ideal of Y, so there is a point $y_0 \in Y$ such that not all $g_i(y_0)$ are zero. Assume without loss of generality that $g_r(y_0) \neq 0$. Then $\sum_{i=1}^{r} g_i(y_0) f_i(x) = 0$ on X, and we have $f_r(x) = \sum_{i=1}^{r-1} [g_i(y_0)/g_r(y_0)] f_i(x)$. Then we get

$$F = \sum_{i=1}^{r-1} f_i \otimes \{g_i(y) + [g_i(y_0)/g_r(y_0)] g_r(y)\},$$

violating the minimality of r. This contradiction shows that ϕ is injective and hence is an isomorphism.

(c)
Let $\pi_X : X \times Y \to X$ be the projection on the first component. Let ϕ be a regular function on X. Then $(\pi_X^*(\phi))(x, y) = (\phi \circ \pi_X)(x, y) = \phi(x)$ is a regular function on $X \times Y$, so π_X is a morphism of varieties. Similarly $\pi_Y : X \times Y \to Y$ is a morphism. Let Z be a variety with morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$. Define $\phi Z \to X \times Y$ as $\phi(z) = (p_X(z), p_Y(z) \in X \times Y$. We then have $\pi_X \circ \phi)(z) = \pi_X(p_X(z), p_Y(z)) = p_X(z)$ and similarly $\pi_Y \circ \phi = p_Y$,

making the given diagram commutative.

This also follows from the universal property of tensor products if we consider the corresponding maps on the coordinate rings.

(d)
This follows from the fact that the dimension of a variety is the Krull dimension of its coordinate ring as follows.

$$\begin{aligned}\dim X \times Y = \dim A(X \times Y) &= \dim A(X) \otimes_k A(Y) \\ &= \dim A(X) + \dim A(Y) = \dim X + \dim Y.\end{aligned}$$

I.3.19

Automorphisms of **$\mathbb{A}^n$. Let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism of $\mathbb{A}^n$ to $\mathbb{A}^n$ given by n polynomials $f_1, \ldots, f_n$ of n variables $x_1, \ldots, x_n$. Let $J = \det |\partial f_i / \partial x_j|$ be the *Jacobian* polynomial of ϕ.**

(a) If ϕ is an isomorphism (in which case we call ϕ an *automorphism*of $\mathbb{A}^n$) show that J is a nonzero constant polynomial.

$**$(b) The converse of (a) is an unsolved problem, even for $n = 2$. See, for example Vitushkin [1].

(a)
If ϕ is invertible, then the Jacobian matrix is also invertible at every point in $\mathbb{A}^n$. Therefore the determinant J of the Jacobian matrix must be nonzero at every point of $\mathbb{A}^n$. But J is a polynomial over an algebraically closed field so will have at least one root unless it is a nonzero constant. Hence J is a nonzero constant.

(b)
This is still an unsolved problem at the time of writing, April 2016.

I.3.20

Let Y be a variety of dimension ≥ 2, and $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show this would be false for $\dim Y = 1$.

See (III, Ex. 3.5) for a generalization.

First we make an observation. Let A be a commutative domain with dimension at least 2. Then we can find two distinct prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ with $0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2$. WE can consider $\mathfrak{p}_1$ as a prime ideal in $A_{\mathfrak{p}_2}$. An element of

$(A_{\mathfrak{p}_2})_{\mathfrak{p}_1}$ is of the form

$$\frac{a/b}{c/d}, \quad \text{where} \quad a,b,c,d \in A,\ b,d \notin \mathfrak{p}_2,\ c \notin \mathfrak{p}_1.$$

But this can be written as

$$\frac{a/b}{c/d} = \frac{ad}{bc}, \quad \text{where now we have} \quad bc \notin \mathfrak{p}_1.$$

This shows that every element of $(A_{\mathfrak{p}_2})_{\mathfrak{p}_1}$ can be considered as an element of $A_{\mathfrak{p}_1}$. Conversely, every element of $A_{\mathfrak{p}_1}$ is of the form

$$\frac{a}{b}, \quad \text{where} \quad a,b \in A,\ b \notin \mathfrak{p}_1.$$

Since $1 \notin \mathfrak{p}_2$, we can rewrite a/b as $(a/1)/(b/1)$ and then it can be considered as an element of $(A_{\mathfrak{p}_2})_{\mathfrak{p}_1}$. Hence the two rings can be considered the same,

$$(A_{\mathfrak{p}_2})_{\mathfrak{p}_1} = A_{\mathfrak{p}_1},$$

which we will use below.

Now let U be any open neighborhood of P in Y, and let $\mathcal{O}(U)$ be the ring of regular functions on U. Let $\mathcal{O}_P$ be the local ring of regular functions at P, with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ be any prime ideal in $\mathcal{O}_P$ with $0 \subset \mathfrak{p} \subset \mathfrak{m}$. Such prime ideals exist since dimension of Y is ≥ 2. There exist a prime ideal $\mathfrak{p}'$ and a maximal ideal $\mathfrak{m}'$ in $\mathcal{O}(U)$ such that $\mathfrak{p} = \mathfrak{p}'\mathcal{O}_P$ and $\mathfrak{m} = \mathfrak{m}'\mathcal{O}_P$. Since $\mathcal{O}_P = \mathcal{O}(U)_{\mathfrak{m}'}$, the above observation gives us the identity

$$(\mathcal{O}_P)_{\mathfrak{p}} = (\mathcal{O}(U)_{\mathfrak{m}'})_{\mathfrak{p}} = \mathcal{O}(U)_{\mathfrak{p}'}.$$

Since P is a normal point, the ring $\mathcal{O}_P$ is a noetherian normal domain. Then we have

$$\mathcal{O}_P = \bigcap_{\substack{\mathfrak{p} \text{ is prime} \\ \text{height } \mathfrak{p}=1}} (\mathcal{O}_P)_{\mathfrak{p}}.$$

This is Theorem 11.5 on page 81 of Matsumura's *Commutative Ring Theory*. Combining this with the previous identity we have

$$\mathcal{O}_P = \bigcap_{\substack{\mathfrak{p}' \text{ is prime and } \mathfrak{p}' \subset \mathfrak{m}' \\ \text{height } \mathfrak{p}'=1}} \mathcal{O}(U)_{\mathfrak{p}'}.$$

Now for any such $\mathfrak{p}'$ choose a point Q on $Z(\mathfrak{p}')$ other than P. Since f is regular at Q, there exists polynomials g and h such that $h(Q) \neq 0$ and $f = g/h$ in some neighborhood of Q. But $h \notin \mathfrak{p}'$ since otherwise it would vanish at Q contrary to our choice of h. This however says that $f = g/h \in \mathcal{O}(U)_{\mathfrak{p}'}$. Since f belongs to every such ring on the right hand side of the above equality, it must belong to $\mathcal{O}_P$, which now means that it is regular at P.

If the dimension of Y is 1, then no prime ideal $\mathfrak{p}$ as above exists so the argument breaks. In fact the conclusion also fails in dimension 1 as the example of $f = 1/x$ on $\mathbb{A}^1 - 0$ shows.

I.7.1 – Final Take-Home Exam

(a) Find the degree of the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ (Ex. 2.12). [Answer: d^n]

(b) Find the degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$ (Ex. 2.14). [Answer: $\binom{r+s}{r}$]

(a)
Let $\phi_d : \mathbb{P}^n \to \mathbb{P}^N$ be the d-uple embedding. Here $N = \binom{n+d}{n} - 1$. Let $P_{n,d}(z)$ be the Hilbert polynomial of $\phi_d(\mathbb{P}^n)$. If z is an integer, then a monomial of degree z in the variables $y_0, \ldots, y_N$ of $\mathbb{P}^N$ pulls back via ϕ_d to a monomial of degree zd in the variables $x_0, \ldots, x_n$ of $\mathbb{P}^n$. But the number of such monomials is known, so

$$P_{n,d}(z) = \binom{n + zd}{n} = \frac{1}{n!} \prod_{\ell=0}^{n-1} (zd + n - \ell) = \frac{d^n}{n!} z^n + \text{lower degree terms in } z.$$

Hence the degree of the d-uple embedding of $\mathbb{P}^n$ is d^n.

Note that $P_{n,d}(0) = 0$, and as we will see in the next exercise, the arithmetic genus of $\phi_d(\mathbb{P}^n)$ is zero.

As a special case consider the d-uple embedding of $\mathbb{P}^1$ into $\mathbb{P}^{d+1}$. Its Hilbert polynomial is $dz + 1$. In particular the Hilbert polynomial of the twisted cubic in $\mathbb{P}^3$ is $3z + 1$.

(b)
Let $\psi_{r,s} : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the Segre embedding. Here $N = rs + r + s$. Let $\Sigma_{r,s} = \psi_{r,s}(\mathbb{P}^r \times \mathbb{P}^s)$. Let $Q_{r,s}(z)$ be the Hilbert polynomial of $\Sigma_{r,s}$. If z is an integer, then a monomial of degree z in the variables $y_0, \ldots, y_N$ of $\mathbb{P}^N$ pulls

back via $\psi_{r,s}$ to a monomials of degree z on $\mathbb{P}^r$ and a monomial of degree z on $\mathbb{P}^s$. But the number of such monomials is known, so

$$\begin{aligned} Q_{r,s}(z) &= \binom{z+r}{r}\binom{z+s}{s} \\ &= \frac{(z+1)(z+2)\cdots(z+r)}{r!}\cdot\frac{(z+1)(z+2)\cdots(z+s)}{s!} \\ &= \frac{1}{r!s!}z^{r+s}+\cdots+1 \\ &= \frac{\frac{(r+s)!}{r!s!}}{(r+s)!}z^{r+s}+\cdots+1 \\ &= \frac{\binom{r+s}{r}}{(r+s)!}z^{r+s}+\cdots+1. \end{aligned}$$

This shows that the degree of $\Sigma_{r,s}$ is $\binom{r+s}{r}$.

I.7.2 – Final Take-Home Exam

Let Y be a variety of dimension r in $\mathbb{P}^n$, with Hilbert polynomial P_Y. We define the *arithmetic genus* of Y to be $p_a(Y) = (-1)^r(P_Y(0) - 1)$. This is an important invariant which (as we will see later in (III, Ex. 5.3)) is independent of the projective embedding of Y.

(a) Show that $p_a(\mathbb{P}^n) = 0$.

(b) If Y is a plane curve of degree d, show that $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

(c) More generally, if H is a hypersurface of degree d in $\mathbb{P}^n$, then $p_a(H) = \binom{d-1}{n}$.

(d) If Y is a complete intersection (Ex. 2.17) of surfaces of degrees a, b in $\mathbb{P}^3$, then $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.

(e) Let $Y^r \subseteq \mathbb{P}^n$, $Z^s \subseteq \mathbb{P}^m$ be projective varieties, and embedd $Y \times Z \subseteq \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ by the Segre embedding. Show that

$$p_a(X \times Z) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z).$$

(a)

The Hilbert polynomial of $\mathbb{P}^n$ is $P_{\mathbb{P}^n}(\ell) = \binom{\ell+n}{n} = \frac{1}{n!}[\ell+1)\cdots(\ell+n)$ which counts the number of distinct monomials of degree ℓ in the variables $x_0, \ldots, x_n$. Since $P_{\mathbb{P}^n}(0) = 1$, the arithmetic genus is 0.

(b)
This is a special case of (c) which we solve below.

(c)
In the proof of Proposition 7.6, page 52, we saw that

$$P_H(z) = \binom{z+n}{n} - \binom{z-d+n}{n}.$$

For the arithmetic genus we need to calculate $P_H(0)$.

$$\begin{aligned} P_H(0) &= \binom{n}{n} - \binom{n-d}{n} \\ &= 1 - \frac{1}{n!}(-d+1)(-d+2)\cdots(-d+n) \\ &= 1 - \frac{(-1)^n}{n!}(d-1)(d-2)\cdots(d-n) \\ &= 1-(-1)^n \; \frac{(d-1)\cdots(d-n)(d-n-1)!}{n!(d-n-1)!} \\ &= 1-(1-)^n\binom{d-1}{n}. \end{aligned}$$

Hence the arithmetic genus is $\binom{d-1}{n} = \frac{(-1)^n}{n!}(d-1)(d-2)\cdots(d-n)$.

Going back to the special case of the (b) part above, putting in $n = 2$ gives the required formula for a plane curve of degree d.

(d)
Let Y_a and Y_b are hypersurfaces in $\mathbb{P}^3$ of degrees a and b respectively with $Y_a = Z(f)$ and $Y_b = Z(g)$ where f and g are homogeneous polynomials of degrees a and b. Assume that $Y = Y_a \cap Y_b$ is a complete intersection, i.e. the two hypersurfaces Y_a and Y_b have no common components.

From the proof of Proposition 7.6, on page 52, we know the Hilbert polynomials of Y_a and Y_b; for example

$$P_{Y_a}(z) = \binom{z+n}{n} - \binom{z-a+n}{n}.$$

We now consider the short exact sequence of graded S-modules

$$0 \longrightarrow (S/f))\,(-b) \xrightarrow{g} S/f \longrightarrow S/(f,g) \longrightarrow 0,$$

where S is the graded polynomial ring $k[x_0, \ldots, x_n]$, the first map is multiplication by g, and the second map is the natural quotient map. Noting that S/f and $S/(f,g)$ are the projective coordinate rings of Y_a and Y respectively, we have from the additivity of the Hilbert function

$$\begin{aligned} P_Y(z) &= P_{Y_a}(z) - P_{Y_a}(z-b) \\ &= \binom{z+3}{3} - \binom{z-a+3}{3} - \left[\binom{z-b+3}{3} - \binom{z-b-a+3}{3}\right]. \end{aligned}$$

For the arithmetic genus we need to put $z = 0$.

$$\begin{aligned} P_Y(0) &= \frac{1}{3!}\left(3! - (-1)^3\left[(a-1)(a-2)(a-3) - (b-1)(b-2)(b-3)\right.\right. \\ &\quad \left.\left. +(a+b-1)(a+b-2)(a+b-3)\right]\right) \\ &= 2ab - \frac{1}{2}a^2b - \frac{1}{2}ab^2. \end{aligned}$$

Hence the arithmetic genus is

$$p_a(Y) = \frac{1}{2}\,ab(a+b-4) + 1.$$

(e)
Let the homogeneous coordinates of $\mathbb{P}^n$ and $\mathbb{P}^m$ be given by $[x_0 : \cdots : x_n]$ and $[y_0 : \cdots : y_m]$. The Segre embedding maps $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^N$, where $N = nm + n + m$, in the following fashion

$$[x_0 : \cdots : x_n] \times [y_0 : \cdots : y_m] \overset{\psi}{\mapsto} [\cdots : x_i y_j : \cdots]$$

where we choose the lexicographical ordering for the placement of the $x_i y_j$. Let the homogeneous coordinates of $\mathbb{P}^N$ be $[\cdots : z_{ij} : \cdots]$ where z_{ij} occupies the place of $x_i y_i$ of the Segre embedding.

For any positive integer r, $z_{ij}^r = x_i^r y_j^r$. It is then clear that any monomial of degree ℓ in the variables z_{ij} is the unique product of a monomial of degree ℓ in the variables x_i with a monomial of degree ℓ in the variables y_j.

The converse works modula $\mathfrak{a}$, where $\mathfrak{a}$ is the ideal of the image of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^N$ under the Segre embedding, see the solution of Ex. I.2.14. This is how it works. Let $p = x_0^{a_0} \cdots x_n^{a_n}$ be a monomial of degree $\ell = a_0 + \cdots + a_n$, and similarly let $q = y_0^{b_0} \cdots y_m^{b_m}$ be another monomial of degree $\ell = b_0 + \cdots + b_m$.

Now we describe an algorithm to construct a monomial M of degree ℓ in the variables z_{ij} starting with the above monomials p and q.

Input p and q as above and initialize $M = 1$.

Let $i \geq 0$ be the smallest index such that $a_i > 0$.
Let $j \geq 0$ be the smallest index such that $b_j > 0$.
Set $c_{ij} = \min\{a_i, b_j\}$.
Replace M by $M \times z_{ij}^{c_{ij}}$.
Replace p by $p/x_i^{c_{ij}}$.
Replace q by $q/y_j^{c_{ij}}$.
Repeat until $p = 1$ or equivalently until $q = 1$.

We thus obtain a monomial of degree ℓ in the variables z_{ij}. Let M' be another monomial of degree ℓ in the variables z_{ij} such that substituting $x_i y_j$ for z_{ij} in M' gives pq. Then $M - M'$ vanishes on the image of the Segre embedding and thus is in $\mathfrak{a}$. Therefore they represent the same monomial in the coordinate ring of $\psi(\mathbb{P}^n \times \mathbb{P}^m)$ which is $k[\ldots, z_{ij}, \ldots]/\mathfrak{a}$.

We conclude that the number of distinct monomials of degree ℓ in the projective coordinate ring of $\psi(\mathbb{P}^n \times \mathbb{P}^m)$ is the product of the number of monomials of degree ℓ on $\mathbb{P}^n$ and $\mathbb{P}^m$. This gives the identity for the Hilbert functions

$$P_{\psi(\mathbb{P}^n \times \mathbb{P}^m)}(z) = P_{\mathbb{P}^n}(z)\, P_{\mathbb{P}^m}(z).$$

Finally returning to our case, we see that the above arguments work for Y, Z and $\psi(Y \times Z)$. Hence we have

$$P_{\psi(Y \times Z)}(z) = P_Y(z)\, P_Z(z).$$

The above algorithm shows that $P_{\psi(Y \times Z)}(z) = P_{Y \times Z}$. Hence finally we have

$$P_{Y \times Z}(z) = P_Y(z)\, P_Z(z).$$

Now we are ready to prove the claim of the exercise. Note that

$$\begin{aligned} p_a(Y \times Z) &= (-1)^{r+s}(P_{Y \times Z}(0) - 1) \\ &= (-1)^{r+s}(P_Y(0) P_Z(0) - 1). \end{aligned}$$

On the other hand we have

$$
\begin{aligned}
&p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z) \\
&\qquad = [(-1)2(P_Y(0) - 1)]\,[(-1)^s(P_Z(0) - 1)] + (-1)[(-1)^s(P_Y(0) - 1)] \\
&\qquad + (-1)^r[(-1)^s(P_Z(0) - 1)] \\
&\qquad = (-1)^{r+s}(P_Y(0)P_Z(0) - 1) \\
&\qquad = p_a(Y \times Z),
\end{aligned}
$$

as claimed.