Bilkent University

Take-Home Exam # 04
Math 633 Algebraic Geometry
Due on: 5 December 2019 Thursday - Class Time
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Q-1) Hartshorne Exercise II.1.13 page 67.
Espace Étale of a presheaf. (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [*Topologie Algébrique et Théorie des Faisceaux* Hermann, Paris (1958), Ch. II, §1.2].) Given a presheaf $\mathcal{F}$ on X, we define a topological space $\text{Spé}(\mathcal{F})$, called the *espace étalé* of $\mathcal{F}$, as follows. As a set $\text{Spé}(\mathcal{F}) = \bigcup_{P \in X} \mathcal{F}_P$. We define a projection map $\pi\colon \text{Spé}(\mathcal{F}) \to X$ by sending $s \in \mathcal{F}_P$ to P. For each open set $U \subseteq X$ and each section $s \in \mathcal{F}(U)$, we obtain a map $\bar{s}\colon U \to \text{Spé}(\mathcal{F})$ by sending $P \mapsto s_P$, its germ at P. This map has the property that $\pi \circ \bar{s} = \text{Id}_U$, in other wors, it is a "section" of π over U. We now make $\text{Spé}(\mathcal{F})$ into a topological space by giving it the strongest topology such that all the maps $\bar{s}\colon U \to \text{Spé}(\mathcal{F})$ for all U, and all $s \in \mathcal{F}(U)$, are continuous. Now show that the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ can be described as follows: for any open set $U \subseteq X$, $\mathcal{F}^+(U)$ is the set of *continuous* sections of $\text{Spé}(\mathcal{F})$ over U. In particular, the original sheaf $\mathcal{F}$ was a sheaf if and only if for each U, $\mathcal{F}(U)$ is equal to the set of all continuous sections of $\text{Spé}(\mathcal{F})$ over U.

Solution:

The key observation here is the following. A subset W of $\text{Spé}(\mathcal{F})$ is open if and only if for every open subset $U \subset X$ and for every $s \in \mathcal{F}(U)$, the set

$$\{x \in U \mid \bar{s}(x) \in W\}$$

is open in X. Then we can see that for any open subset U in X, and for any section $s \in \mathcal{F}(U)$ the sets

$$(s, U) = \{s_x \in \mathcal{F}_x \mid x \in U\} \subset \bigsqcup_{x \in U} \mathcal{F}_x.$$

form a basis for a topology of open sets on $\text{Spé}(\mathcal{F})$. With this topology, $\bar{s} : U \to \text{Spé}(\mathcal{F})$ is continuous: For any $s_x \in (s, U)$, an open neighbourhood of s_x is of the form (s, V) where V is an open neighbourhood of x contained in V. Then $\bar{s}^{-1}((s, V)) = V$, hence $\bar{s}^{-1}$ pulls back open sets to open sets and thus is continuous.

Now undoing the definition of $\mathcal{F}^+$ shows that we obtain all the continuous sections of $\text{Spé}(\mathcal{F})$.