Bilkent University

Take-Home Exam # Final
Math 633 Algebraic Geometry
Due on: 10 January 2010 Friday
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Solution Key

Q-1) In Griffiths & Harris, *Principles of Algebraic Geometry*, on page 39, during the definition of the Čech cohomology, the authors claim that the induced map ρ on cohomology is independent of the choice ϕ. Prove this claim.

This is intended to give you a realistic taste of mathematics: you have to do some serious calculations before you say something simple as "the result is independent of the choice made".

Hint: You will find a reasonable calculation of this as the proof of Lemma 3.2 on page 117 in *Complex Manifolds and Deformation of Complex Structures*, by Kunihiko Kodaira, Springer-Verlag (1986). Even though the line of proof is correct, there are a few typos in the proof and some unexplained stages in the calculation. Your mission is to provide those missing explanations and carry out the calculations fully.

Solution:

We will follow Kodaira's proof, as cited above, and explain the missing steps.

Let M be a topological space, and $\mathcal{F}$ a sheaf on M.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of M and $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ a refinement of $\mathcal{U}$. This means that there exists a map $k\colon \Lambda \to I$ such that for every $\lambda \in \Lambda$ we have $V_\lambda \subseteq U_{k(\lambda)}$.

If $\sigma = \{\sigma_{i_0\cdots i_q}\} \in C^q(\mathcal{U}, \mathcal{F})$, then we can define $\rho_k\sigma = \{(\rho_k\sigma)_{\lambda_0\cdots\lambda_q}\} \in C^q(\mathcal{V}, \mathcal{F})$ where

$$(\rho_k\sigma)_{\lambda_0\cdots\lambda_q} = r_V \sigma_{k(\lambda_0)\cdots k(\lambda_q)},$$

where $V = \bigcup_{s=0}^{q} V_{\lambda_t}$ and

$$r_V : \mathcal{F}(U_{k(\lambda_0)} \cap \cdots \cap U_{k(\lambda_q)}) \to \mathcal{F}(V)$$

is the usual restriction map. Thus k induces a map

$$\rho_k\colon C^q(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{V}, \mathcal{F}),$$

Since the map ρ_k has the property $\delta \circ \rho_k = \rho_k \circ \delta$, it also induces a map

$$\tilde{\rho_k} : \check{H}^q(\mathcal{U}, \mathcal{F}) \to \check{H}^q(\mathcal{V}, \mathcal{F}).$$

Now let $j : \Lambda \to I$ be another map such that for every$\lambda \in \Lambda$ we have $V_\lambda \subseteq U_{j(\lambda)}$.

We want to prove that $\tilde{\rho_k} = \tilde{\rho_j}$.

To show that $\tilde{\rho_k} = \tilde{\rho_j}$, clearly it suffices to show that for any $\sigma \in Z^q(\mathcal{U}, \mathcal{F})$, the equality $\rho_j\sigma - \rho_k\sigma \in \delta C^{q-1}(\mathcal{V}, \mathcal{F})$ holds.

Let $\eta_{\lambda_0\cdots\lambda_q} = r_V\sigma_{j(\lambda_0)\cdots j(\lambda_q)}$ and $\tau_{\lambda_0\cdots\lambda_q} = r_V\sigma_{k(\lambda_0)\cdots k(\lambda_q)}$. We will define an element $\tilde{\kappa} \in C^{q-1}(\mathcal{V}, \mathcal{F})$ such that

$$\{\eta_{\lambda_0\cdots\lambda_q}\} - \{\tau_{\lambda_0\cdots\lambda_q}\} = (\delta\tilde{\kappa})_{\lambda_0\cdots\lambda_q}. \qquad (\spadesuit)$$

We first note that for any $t = 1, \ldots, q$, and any $\nu_1, \ldots, \nu_q \in \Lambda$, we have we have

$$\sigma_{k(\nu_1)\cdots k(\nu_t)j(\nu_t)\cdots j(\nu_q)} \in \mathcal{F}(U_{k(\nu_1)} \cap \cdots \cap U_{k(\nu_t)} \cap U_{j(\nu_t)} \cap \cdots \cap U_{j(\nu_q)}).$$

Moreover we have

$$V_{\nu_1} \subseteq U_{k(\nu_1)}, \ldots, V_{\nu_t} \subseteq U_{k(\nu_t)}, V_{\nu_t} \subseteq U_{j(\nu_t)}, \ldots, V_{\nu_q} \subseteq U_{j(\nu_q)},$$

and since V_{ν_t} is repeated, if we set $W = \bigcap_{t=1}^{q} V_{\nu_t}$, then

$$W \subseteq U_{k(\nu_1)} \cap \cdots \cap U_{k(\nu_t)} \cap U_{j(\nu_t)} \cap \cdots \cap U_{j(\nu_q)},$$

and the restriction map

$$r_W : \mathcal{F}(U_{k(\nu_1)} \cap \cdots \cap U_{k(\nu_t)} \cap U_{j(\nu_t)} \cap \cdots \cap U_{j(\nu_q)}) \to \mathcal{F}(W)$$

is well defined. Now set

$$\kappa_{\nu_1\cdots\nu_q} = \sum_{t=1}^{q} (-1)^{t-1} r_W \sigma_{k(\nu_1)\cdots k(\nu_t)j(\nu_t)\cdots j(\nu_q)}.$$

We define

$$(\delta\kappa)_{\lambda_0\cdots\lambda_q} = \sum_{s=0}^{q} (-1)^s r_V \kappa_{\lambda_0\cdots\widehat{\lambda_s}\cdots\lambda_q}.$$

We now claim that

$$(\delta\kappa)_{\lambda_0\cdots\lambda_q} = \eta_{\lambda_0\cdots\lambda_q} - \tau_{\lambda_0\cdots\lambda_q}. \qquad (*)$$

During the proof of this claim for ease of notation we write h_s for $k(\lambda_s)$, and j_s for $j(\lambda_s)$.

$$\begin{aligned}
(\delta\kappa)_{\lambda_0\cdots\lambda_q} &= \sum_{s=0}^{q} (-1)^s r_V \kappa_{\lambda_0\cdots\widehat{\lambda_s}\cdots\lambda_q} \\
&= \sum_{s=0}^{q} (-1)^s \left(\sum_{t=0}^{s-1} (-1)^t r_V \sigma_{h_0\cdots h_t j_t \cdots \widehat{j_s}\cdots j_q} + \sum_{t=s+1}^{q} r_V \sigma_{h_0\cdots\widehat{h_s}\cdots h_t j_t\cdots j_q} \right) \\
&= \sum_{t=0}^{q} (-1)^{t+1} \left(\sum_{s=0}^{t-1} (-1)^s r_V \sigma_{h_0\cdots\widehat{h_s}\cdots h_t j_t\cdots j_q} + \sum_{s=t+1}^{q} (-1)^{s+1} r_V \sigma_{h_0\cdots h_t j_t\cdots\widehat{j_s}\cdots j_q} \right) \\
&= \sum_{t=0}^{q} (-1)^{t+1} \left(\sum_{s=0}^{t} (-1)^s r_V \sigma_{h_0\cdots\widehat{h_s}\cdots h_t j_t\cdots j_q} + \sum_{s=t}^{q} (-1)^{s+1} r_V \sigma_{h_0\cdots h_t j_t\cdots\widehat{j_s}\cdots j_q} \right) \\
&\quad - \sum_{t=0}^{q} (-1)^{t+1} \left((-1)^t r_V \sigma_{h_0\cdots\widehat{h_t} j_t\cdots j_q} + (-1)^{t+1} r_V \sigma_{h_0\cdots h_t \widehat{j_t}\cdots j_q} \right) \\
&= \sum_{t=0}^{q} (-1)^{t+1} \left((\delta\sigma)_{h_0\cdots h_t j_t\cdots j_q} \right) + \sum_{t=0}^{q} (r_V \sigma_{h_0\cdots\widehat{h_t} j_t\cdots j_q} - r_V \sigma_{h_0\cdots h_t \widehat{j_t}\cdots j_q}) \qquad (\heartsuit) \\
&= \eta_{\lambda_0\cdots\lambda_q} - \tau_{\lambda_0\cdots\lambda_q},
\end{aligned}$$

where we notice that the last summation is a telescoping sum, and recall the assumption that $\delta\sigma = 0$.

The $\kappa_{\nu_1\cdots\nu_q}$ that we used is not skew symmetric in its indices so we define

$$\tilde{\kappa}_{\nu_1\cdots\nu_q} = \frac{1}{q!}\sum \operatorname{sgn}\begin{pmatrix}\nu_1\cdots\nu_q\\ \mu_1\cdots\mu_q\end{pmatrix}\kappa_{\mu_1\cdots\mu_q},$$

where the summation is over all permutations of $\nu_1,\ldots,\nu_q$.

Thus $\tilde{\kappa} = \{\tilde{\kappa}_{\nu_1\cdots\nu_q}\} \in C^{q-1}(\mathcal{V})$.

Since $\eta_{\lambda_0\cdots\lambda_q}$ and $\tau_{\lambda_0\cdots\lambda_q}$ are skew-symmetric on their indices, and from equation $(*)$ we have

$$\begin{aligned}\eta_{\lambda_0\cdots\lambda_q} - \tau_{\lambda_0\cdots\lambda_q} &= \frac{1}{(q+1)!}\sum \operatorname{sgn}\begin{pmatrix}\lambda_0\cdots\lambda_q\\ \nu_0\cdots\nu_q\end{pmatrix}(\eta_{\nu_0\cdots\nu_q} - \tau_{\nu_0\cdots\nu_q})\\ &= \frac{1}{(q+1)!}\sum \operatorname{sgn}\begin{pmatrix}\lambda_0\cdots\lambda_q\\ \nu_0\cdots\nu_q\end{pmatrix}(\delta\kappa)_{\nu_0\cdots\nu_q}\\ &= \frac{1}{(q+1)!}\sum \operatorname{sgn}\begin{pmatrix}\lambda_0\cdots\lambda_q\\ \nu_0\cdots\nu_q\end{pmatrix}\sum_{s=0}^{q}(-1)^s\kappa_{\nu_0\cdots\widehat{\nu_s}\cdots\nu_q},\end{aligned}$$

where the summations are taken over all permutations of $\lambda_0,\ldots,\lambda_q$.

We rewrite the last line as a sum $A_0 + \cdots + A_q$ where

$$A_i = \frac{1}{(q+1)!}\sum_{s=0}^{q}(-1)^s\sum_{\nu_s=\lambda_i}\operatorname{sgn}\begin{pmatrix}\lambda_0\cdots\lambda_q\\ \nu_0\cdots\nu_q\end{pmatrix}\kappa_{\nu_0\cdots\widehat{\nu_s}\cdots\nu_q},$$

where we considered only those permutations of $\lambda_0,\ldots,\lambda_q$ into $\nu_0,\ldots,\nu_q$ with $\nu_s = \lambda_i$, so that the omitted index of κ is always λ_i.

To better understand the nature of each A_i we undo A_0 first. Let $\{\lambda_0,\mu_1,\ldots,\mu_q\} = \{\lambda_0,\lambda_1,\ldots,\lambda_q\}$. Then a permutation as in the summation of A_0 is of the form

$$\begin{pmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{s-1} & \lambda_s & \lambda_{s+1} & \cdots & \lambda_q\\ \mu_1 & \mu_2 & \cdots & \mu_s & \lambda_0 & \mu_{s+1} & \cdots & \mu_q\end{pmatrix},$$

which we can write as the composition of two permutations

$$\begin{pmatrix}\lambda_0 & \mu_1 & \cdots & \mu_{s-1} & \mu_s & \mu_{s+1} & \cdots & \mu_q\\ \mu_1 & \mu_2 & \cdots & \mu_s & \lambda_0 & \mu_{s+1} & \cdots & \mu_q\end{pmatrix}\circ\begin{pmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{s-1} & \lambda_s & \lambda_{s+1} & \cdots & \lambda_q\\ \lambda_0 & \mu_1 & \cdots & \mu_{s-1} & \mu_s & \mu_{s+1} & \cdots & \mu_q\end{pmatrix},$$

This gives

$$\operatorname{sgn}\begin{pmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{s-1} & \lambda_s & \lambda_{s+1} & \cdots & \lambda_q\\ \mu_1 & \mu_2 & \cdots & \mu_s & \lambda_0 & \mu_{s+1} & \cdots & \mu_q\end{pmatrix} = (-1)^s\operatorname{sgn}\begin{pmatrix}\lambda_1 & \cdots & \lambda_q\\ \mu_1 & \cdots & \mu_q\end{pmatrix}$$

Hence we will have

$$\begin{aligned} A_0 &= \frac{1}{(q+1)!}\sum_{s=0}^{q}(-1)^s \sum_{\nu_s=\lambda_0} \operatorname{sgn}\begin{pmatrix}\lambda_0\cdots\lambda_q\\ \nu_0\cdots\nu_q\end{pmatrix}\kappa_{\nu_0\cdots\widehat{\nu_s}\cdots\nu_q} \\ &= \frac{1}{(q+1)!}\sum_{s=0}^{q}(-1)^s \sum_{\substack{\text{Permutations}\\ \text{on } \lambda_1,\ldots,\lambda_q}} (-1)^s \operatorname{sgn}\begin{pmatrix}\lambda_1\cdots\lambda_q\\ \mu_1\cdots\mu_q\end{pmatrix}\kappa_{\mu_1\cdots\mu_q} \\ &= \frac{1}{q!}\sum_{\substack{\text{Permutations}\\ \text{on } \lambda_1,\ldots,\lambda_q}} \operatorname{sgn}\begin{pmatrix}\lambda_1\cdots\lambda_q\\ \mu_1\cdots\mu_q\end{pmatrix}\kappa_{\mu_1\cdots\mu_q} \\ &= \tilde{\kappa}_{\mu_1\cdots\mu_q}. \end{aligned}$$

A similar calculation will give, for $i = 0, \ldots, q$,

$$A_i = (-1)^i \tilde{\kappa}_{\mu_0\cdots\widehat{\mu_i}\cdots\mu_q}.$$

Since $\{\mu_0, \ldots, \mu_q\} = \{\lambda_0, \ldots, \lambda_q\}$ and since $\sum_{i=0}^{q}(-1)^i\tilde{\kappa}_{\mu_0\cdots\widehat{\mu_i}\cdots\mu_q}$ is skew-symmetric in its indices, we get

$$\begin{aligned} \eta_{\lambda_0\cdots\lambda_q} - \tau_{\lambda_0\cdots\lambda_q} &= A_0 - A_1 + \cdots + (-1)^q A_q \\ &= \sum_{i=0}^{q}(-1)^i\tilde{\kappa}_{\mu_0\cdots\widehat{\mu_i}\cdots\mu_q} \\ &= \sum_{i=0}^{q}(-1)^i\tilde{\kappa}_{\lambda_0\cdots\widehat{\lambda_i}\cdots\lambda_q} \\ &= (\delta\widehat{\kappa})_{\lambda_0\cdots\lambda_q}. \end{aligned}$$

This finally establishes equation (♠) which was our aim in this note.

Remark: The first summation in equation (♡) was zero since we chose σ as a cocycle. If instead we chose σ as a cochain only, and if we denote by $h : C^q(\mathcal{U}, \mathcal{F}) \to C^{q-1}(\mathcal{V}, \mathcal{F})$ the process of obtaining $\tilde{\kappa}$ from σ, then the above calculations would show that we have

$$\rho_j - \rho_k = \delta h + h\delta,$$

which says that ρ_j and ρ_k are chain homotopic, and hence induce the same map in cohomology. This is in fact what was asked from the reader in Griffiths and Harris's book on page 39. ☺

$$\rho_\varphi : C^p(\underline{U}, \mathfrak{F}) \to C^p(\underline{U}', \mathfrak{F})$$

given by

$$(\rho_\varphi\sigma)_{\beta_0\cdots\beta_p} = \sigma_{\varphi\beta_0\cdots\varphi\beta_p}|_{U'_{\beta_0}\cap\cdots\cap U'_{\beta_p}}$$

Evidently $\delta\circ\rho_\varphi = \rho_\varphi\circ\delta$, and so ρ_φ induces a homomorphism

$$\rho : H^p(\underline{U}, \mathfrak{F}) \to H^p(\underline{U}', \mathfrak{F}),$$

which is independent of the choice of φ. (The reader may wish to check that the chain maps ρ_φ and ρ_ψ associated to two inclusion associations φ and ψ are *chain homotopic* and thus induce the same map on cohomology.) We define the p[th] *Čech cohomology group* of $\mathfrak{F}$ on M to be the direct limit of the $H^p(\underline{U}, \mathfrak{F})$'s as $\underline{U}$ becomes finer and finer:

$$H^p(M, \mathfrak{F}) = \varinjlim_{U} H^p(\underline{U}, \mathfrak{F}).$$