An Algebraic Interpretation of the Multiplicity Sequence of an Algebraic Branch

Une Interprétation Algébrique de la Suite
des Ordres de Multiplicité d'une
Branche Algébrique

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The sequence of multiplicities of successive points of an algebraic branch can be defined through purely algebraic notions. In what follows we present such a definition which does not differ from the geometric definition except by its form. We hope that this definition will constitute an answer to a question posed by P. Du Val* on the relation which exists between his results and the power series[2] expansion of the branch under consideration. *(p 256)*[1]

Section 1: [3]

k being any field, we consider a ring H formed by some power series of a single variable t with coefficients in k. Let

$$W(H) = \{i_0 = 0, i_1, i_2, \dots, i_r, i_{r+1}, \dots\}^4$$

*P. Du Val, "The Jacobian algorithm and the multiplicity sequence of an algebraic branch", *Rev. Faculté Sci. Univ. Istanbul* (Série A), 7 (1942), 107-112.

be the orders (i.e. the degrees of the first terms with non-zero coefficients) of the elements of H. The integers $i_0, i_1, \ldots, , i_r, \ldots$ form a semigroup of the non-negative[5] integers. $S_0, S_{i_1}, S_{i_2}, \ldots, S_{i_r}, \ldots$ being elements of H of orders $i_0, i_1, \ldots, i_r, \ldots$ respectively, any element of this ring is of the form

$$\sum_{\ell=0}^{\infty} \alpha_\ell S_{i_\ell} \quad (\alpha_\ell \in k).$$

We assume that H contains all the series of this form. We denote by I_h the set of elements of H of orders larger than or equal to h. I_h is clearly an ideal of H and its elements are of the form

$$\sum_{i_\ell \geq h}^{\infty} \alpha_\ell S_{i_\ell} \quad (\alpha_\ell \in k).$$

Lemma 1. [6] *ν being the* $\gcd$ *of the elements of $W(H)$, for r sufficiently large, one has* *(p 257)*

$$i_{r+1} = i_r + \nu, i_{r+2} = i_r + 2\nu, \ldots, i_{r+\ell} = i_r + \ell\nu, \ldots$$

and there exists a power series of order 1,

$$\tau = t\left(1 + \sum_{\ell=1}^{\infty} \delta_\ell t^\ell\right) \quad (\delta_\ell \in k)$$

such that every element of H is of the form $\sum_{j=0}^{\infty} \alpha_j \tau^{j\nu}$.

Proof. Let us denote the $\gcd$ of the integers $i_1, i_2, \ldots, i_\ell$ by ν_ℓ. Each of these numbers divides all those that come before it. It follows that for ρ sufficiently large we have $\nu_\rho = \nu_{\rho+1} = \nu_{\rho+2} = \cdots = \nu$. Then let

$$\nu = m_1 i_1 + m_2 i_2 + \cdots + m_\rho i_\rho,$$

$m_1, m_2, \ldots, m_\rho$ being integers which are positive, zero or negative. m being the largest of the integers $|m_h(i_1/\nu - 1)|$, the multiples of ν which are greater than

$$i = m i_1 + m i_2 + \cdots + m i_\rho$$

are contained in $W(H)$. In fact we have, for $\ell = 0, 1, 2 \ldots, i_1/\nu - 1$,

$$\begin{aligned} i + \ell\nu &= (m + \ell m_1) i_1 + (m + \ell m_2) i_2 + \cdots + (m + \ell m_\rho) i_\rho \\ &= n_1 i_1 + n_2 i_2 + \cdots + n_\rho i_\rho, \end{aligned}$$

with $n_h \geq 0$; since $m \geq |m_h \ell|$. For $\ell = i_1/\nu$, we have $i + i_1 \in W(H)$. In general, the multiples of ν which are greater than i can be written in the form $i + ji_1 + \ell\nu$ $(\ell = 0, 1, 2, \ldots, i_1/\nu - 1,\ j \geq 0)$ and it is obvious that all of these integers are of the form $\sum_{h=1}^{\rho} n_h i_h$ with $n_h \geq 0$; i.e. belong to $W(H)$. $S_{i_1} = \sum_{\ell=i_1}^{\infty} \sigma_\ell t^\ell$ $(\sigma_\ell \in k,\ \sigma_{i_1} \neq 0)$[7] being an element of order i_1 in H, we can choose a power series of the form $\tau = t\left(1 + \sum_{\ell=1}^{\infty} \delta_\ell t^\ell\right)$, $(\delta_\ell \in k)$ in such a way that we have $S_{i_1} = \sigma_{i_1} \tau^{i_1}$. Under these conditions the power series in t with coefficients in k may be written in the form of power series in τ with coefficients in k. *In particular the elements of H can be written in the form* $\sum_{j=0}^{\infty} \alpha_{j\nu} \tau^{j\nu}$. It suffices to prove this for the elements of H of orders greater than i; since every element of H can be considered as a quotient of an element in H of order greater than i by a suitably chosen power of $S_{i_1} = \sigma_{i_1} \tau^{\nu(i_1/\nu)}$. The orders of the elements of H being multiples of ν, any element of H is of the form $\sum_{j=N\nu}^{\infty} \alpha_j \tau^j$ $(\alpha_j \in k,\ \alpha_{N\nu} \neq 0)$. For $N\nu \geq i$, the ring H contains the elements, $S_{N\nu+\nu}, S_{N\nu+2\nu}, \ldots, \left(S_{N\nu+\ell\nu} = \sum_{j=N\nu+\ell\nu}^{\infty} \alpha_{\ell,j} \tau^j,\ \alpha_{\ell,j} \in k,\ \alpha_{\ell,N\nu+\ell\nu} \neq 0\right)$ *(p 258)* of orders $N\nu + \nu, N\nu + 2\nu, \ldots$ respectively. We can then choose the series $\sum_{\ell=1}^{\infty} \beta_\ell S_{N\nu+\ell\nu}$ in such a way that the difference

$$S_{N\nu} = \sum_{j=N\nu}^{\infty} \alpha_j \tau^j - \sum_{\ell=1}^{\infty} \beta_\ell S_{N\nu+\ell\nu} = \alpha_{N\nu} \tau^{N\nu} + \tilde{\alpha}_\mu \tau^\mu + \cdots$$

does not contain any terms of order divisible by ν, other than the first. Indeed suppose that $\beta_1, \beta_2, \ldots, \beta_h$ are chosen such that the terms of orders $N\nu + \nu, N\nu + 2\nu, \ldots, N\nu + h\nu$ of the difference

$$\sum_{j=N\nu}^{\infty} \alpha_j \tau^j - \sum_{\ell=1}^{h} \beta_\ell S_{N\nu+\ell\nu} = \alpha_{N\nu} \tau^{N\nu} + \alpha_{\mu_h}^{(h)} \tau^{\mu_h} + \cdots$$

vanish; it suffices then to set

$$\beta_{h+1} = \frac{\alpha^{(h)}_{N\nu+h\nu+\nu}}{\alpha_{h+1,N\nu+h\nu+\nu}}$$

so that the terms of orders $N\nu+\nu, N\nu+2\nu, \ldots, N\nu+h\nu, N\nu+h\nu+\nu$ of the difference

$$\sum_{j=N\nu}^{\infty} \alpha_j \tau^j - \sum_{\ell=1}^{h+1} \beta_\ell S_{N\nu+\ell\nu} = \alpha_{N\nu}\tau^{N\nu} + \alpha^{(h+1)}_{\mu_{h+1}} \tau^{\mu_{h+1}} + \cdots$$

vanish. Under these conditions the series $S_{N\nu}$ reduces to $\alpha_{N\nu}\tau^{N\nu}$. Otherwise the difference

$$S_{N\nu}^{i_1/\nu} - \alpha_{N\nu}^{i_1/\nu} \left(\frac{S_{i_1}}{\sigma_{i_1}}\right)^N = \frac{i_1}{\nu} \alpha_{N\nu}^{i_1/\nu-1} \tilde{\alpha}_\mu \tau^{N\nu(i_1/\nu-1)+\mu} + \cdots$$

whose order is not divisible by ν will be in H. Therefore every element of H of order greater then i is a linear combination with coefficients in k of elements of the form $\alpha_{N\nu}\tau^{N\nu} = S_{N\nu}$. □[8]

Remark. After the preceding theorem, the ring H may be considered as a subring of the ring of power series of the variable $T = \tau^\nu$ with coefficients in k. Let us set ${}^*i_h = i_h/\nu$. The orders of the elements of H with respect to this new variable will be ${}^*i_0 = 0, {}^*i_1, {}^*i_2, \ldots, {}^*i_r, \ldots$, and for r sufficiently large, one will have

$${}^*i_{r+1} = {}^*i_r + 1, {}^*i_{r+2} = {}^*i_r + 2, \ldots.$$

Lemma 2. *The inverse of every element of order zero of H is also an element of H.*

Proof. If the order of $a = \sum_{h=0}^{\infty} \alpha_h S_{i_h}$ is zero, then α_0 is different than zero. In fact the coefficients β_h of the product

$$\alpha_0^{-1} \prod_{h=1}^{\infty} (1 + \beta_h S_{i_h})$$

can be chosen such that we have *(p 259)*

$$a\alpha_0^{-1}\prod_{h=1}^{n}(1+\beta_h S_{i_h}) \equiv 1 \mod t^{i_n+1}.$$

Suppose now that this choice has been made for $\beta_1, \beta_2, \dots, \beta_{n-1}$. We have

$$a\alpha_0^{-1}\prod_{h=1}^{n-1}(1+\beta_h S_{i_h}) = 1 + \gamma_n S_{i_n} + \gamma_{n+1} S_{i_{n+1}} + \cdots$$

and it suffices to set $\beta_n = -\gamma_n$ to have

$$a\alpha_0^{-1}\prod_{h=1}^{n}(1+\beta_h S_{i_h}) \equiv 1 \mod t^{i_n+1}.$$

For the coefficients β_h chosen in this manner we obviously have

$$a\alpha_0^{-1}\prod_{h=1}^{\infty}(1+\beta_h S_{i_h}) = 1.$$

□

Remark. $\sum_{h=0}^{\infty} \alpha_h S_{i_h}$ being an element of order zero in H, to each n-th root of α_0 contained in k corresponds an n-th root of $\sum_{h=0}^{\infty} \alpha_h S_{i_h}$ contained in H. The proof of this fact is similar to that of Lemma 2.

Section 2:

Lemma 3. *If one denotes by I_h/S_h the set of quotients of elements of I_h by S_h, and by $[I_h/S_h]$ the ring generated by I_h/S_h, the ring $[I_h/S_h]$ does not depend on the choice of S_h among the elements of H of order h.*

Proof. Let us first note that the set I_h/S_h contains the ring H and consequently $[I_h/S_h] \supseteq H$.

Let $S'_h = \epsilon S_h$ be another element of order h in H. ϵ is then an element of $[I_h/S_h]$. It follows from Lemma 2 that ϵ^{-1} is also an element of $[I_h/S_h]$. We then have

$$I_h/S'_h = I_h/\epsilon S_h = \epsilon^{-1}(I_h/S_h) \subseteq [I_h/S_h]$$

and therefore

$$[I_h/S'_h] \subseteq [I_h/S_h].$$

We can obviously show in exactly the same manner that we also have

$$[I_h/S_h] \subseteq [I_h/S'_h].$$

We then have $[I_h/S'_h] = [I_h/S_h]$. □

The ring $[I_h/S_h]$ being independent of the choice of S_h, we can denote it by $[I_h]$.

Remark. The semigroup $W([I_h])$ clearly contains the semigroup generated by the integers

$$i_h - i_h = 0, i_{h+1} - i_h, i_{h+2} - i_h, \dots$$

which are the orders of the elements of I_h/S_{i_h}. But as the following example shows, $W([I_{i_h}])$ is not necessarily equal to this semigroup: *(p 260)*

Let us consider the ring H formed by all series of the form

$$\sum_{i,j\geq 0} \alpha_{ij} X^i Y^j \quad (\alpha_{ij} \in k),$$

where $X = t^4$, $Y = t^{10} + t^{15}$. One easily shows that $W(H)$ is formed by the integers

$$0,4, 8, 10, 12, 14, 16, 18, 20, 22, 24, 25, 26$$
$$28, 29, 30, 32, 33, 34, 35, 36, 37, 38, \dots.$$

Then the orders of the elements of I_4/X are the integers

$$0, 4, 6, 8, 10, 12, 14, 16, 18, 20, 21, 22,$$
$$24, 25, 26, 28, 29, 30, 31, 32, 33, 34, \dots$$

which generate the semigroup

$$0,4, 6, 8, 10, 12, 14, 16, 18, 20, 21, 22, 24,$$
$$25, 26, 27, 28, 29, 30, 31, 32, 33, 34, \dots$$

while $[I_4]$ contains the element $(Y/X)^2 - X^2 = 2t^{17} + t^{22}$ whose order is 17.

Remark. If for some particular choice of S_h, the ring $[I_h]$ is equal to I_h/S_h, then it is the same for all choices of S_h. In fact, $S'_h = \epsilon S_h$ being another element of order h of H, we have

$$I_h/S'_h = \epsilon^{-1} I_h/S_h = \epsilon^{-1}[I_h] = [I_h];$$

since every element S of $[I_h]$ is equal to an element ϵS of $[I_h]$ multiplied by ϵ^{-1}.

Definition. *We say that the ring H is canonical*[9] *if one has $[I_h] = I_h/S_h$ for all $h \in W(H)$.*

Remark. If H is a canonical ring, the integers

$$i_h - i_h = 0, i_{h+1} - i_h, i_{h+2} - i_h, \dots$$

form a semigroup for every h. *A semigroup of non-negative integers*

$$i_0 = 0, i_1, i_2, \dots, i_h, \dots$$

is called canonical if the sequence

$$i_h - i_h = 0, i_{h+1} - i_h, i_{h+2} - i_h, \dots$$

is a semigroup for each h. If the sequence of increasing integers

$$i_0 = 0, i_1, i_2, \dots, i_h, \dots$$

is a canonical semigroup, then the power series

$$\sum_{h=0}^{\infty} \alpha_h t^{i_h} \quad (\alpha_h \in k),$$

clearly forms a canonical ring. $W(H)$ *can be canonical without it being the case for H*: The ring formed by the series of the form $\sum_{i,j,\ell \geq 0} \alpha_{ij\ell} X^i Y^j Z^\ell$ $(\alpha_{ij\ell} \in k)$, with $X = t^4, Y = t^{10} + t^{15}, Z = t^{27}$, is such that the orders *(p 261)*

$$0, 4, 8, 10, 12, 14, 16, 18, 20, 22, 24, 25, 26, 27, 28, 29, 30, \dots$$

of its elements form, as one can easily verify, a canonical semigroup, while H is not a canonical ring, since $[I_4]$ an element of order 17, $(Y/X)^2 - X^3 = 2t^{17} + t^{22}$, which is not contained in I_4/X.

Lemma 4. *The intersection of several canonical rings is canonical.*

Proof. It obviously suffices to prove the lemma only for the intersection of two canonical rings. H and H' being two canonical rings, let S be a common element of these two rings. Let h be the order of S let I_h and I'_h be the set of elements of H and H' whose orders are not less than h. It suffices to show that

$$(I_h \cap I'_h)/S = I_h/S \cap I'_h/S$$

is a ring. Now I_h/S and I'_h/S being rings, it is the same for their intersection. $\square$

Remark. If H is a canonical ring, then so is $[I_{i_h}]$. Indeed consider the set of elements of I_{i_h}. These elements are of the form

$$\sum_{\nu=h}^{\infty} \alpha_\nu S_{i_\nu} \quad (\alpha_\nu \in k).$$

H being a canonical ring, the ring $[I_{i_h}]$ consists of the set of series of the form $\sum_{\nu=h}^{\infty} \alpha_\nu \frac{S_{i_\nu}}{S_{i_h}}$ whose orders are the numbers

$$0, j_1 = i_{h+1} - i_h, j_2 = i_{h+2} - i_h, \ldots.$$

The set of elements of $[I_{i_h}]$ of order greater than or equal to j_ℓ is then the set of series of the form

$$\sum_{\nu=h+\ell}^{\infty} \alpha_\nu \frac{S_{i_\nu}}{S_{i_h}} \quad (\alpha_\nu \in k).$$

$S_{i_{h+\ell}}/S_{i_h}$ being an element of order $j_\ell = i_{h+\ell} - i_h$ of this set, the set of elements

$$\left(\sum_{\nu=h+\ell}^{\infty} \alpha_\nu \frac{S_{i_\nu}}{S_{i_h}}\right) \Bigg/ \frac{S_{i_{h+\ell}}}{S_{i_h}} = \sum_{\nu=h+\ell}^{\infty} \alpha_\nu \frac{S_{i_\nu}}{S_{i_{h+\ell}}}$$

is the ring $[I_{i_{h+\ell}}]$.

$\mathbb{N}$ being the set of all non-negative integers*, we show in a similar manner that *(p 262)* if

$$\{0, i_1, i_2, \ldots, i_r + \mathbb{N}\nu\}$$

*In what follows $\mathbb{N}$ will always denote the set of all non-negative integers.[10]

is a canonical semigroup, it is the same for

$$\{0, i_{h+1} - i_h, \ldots, i_r - i_h + \mathbb{N}\nu\}.$$

Remark. If the integers

$$i_0 = 0, i_1, i_2, \ldots, i_h, \ldots$$

form a canonical semigroup, then we have $i_{h+1} - i_h \leq i_h - i_{h-1}$. In fact, before the integers $i_{h-1} - i_{h-1} = 0, i_h - i_{h-1}, i_{h+1} - i_{h-1}, \ldots, i_r - i_{h-1}, \ldots$ form a semigroup, we must have $i_{h+1} - i_{h-1} \leq 2(i_h - i_{h-1})$; from which the inequality $i_{h+1} - i_h \leq i_h - i_{h-1}$ follows immediately.

Section 3:

From the remark which follows immediately Lemma 1, $I_{(N-1)\nu}$ contains all the power series whose orders in $T = \tau^\nu$ are greater than or equal to $N - 1$ provided that N is sufficiently large. $[I_{(N-1)\nu}]$ is then the ring $k[T]$ of all the power series in T with coefficients in k. This remark leads to the following construction which allows us to obtain all the canonical rings as well as all the canonical semigroups.

We begin by considering the ring $[I_{(N-1)\nu}] = k[T]$ of all power series in T and the semigroup $\mathbb{N}\nu$ of multiples of ν by non-negative integers. We choose an element T_{r-1} of non-zero order in $[I_{(N-1)\nu}]$, and a non-zero element $\nu_{r-1}(= w(T_{r-1})^*)$ in $\mathbb{N}\nu$ and we set

$$[I_{i_{r-1}}] = k + T_{r-1}[I_{(N-1)\nu}] \quad (i_r = (N-1)\nu).$$

The ring $[I_{i_{r-1}}]$ and the semigroup $\{0, \nu_{r-1} + \mathbb{N}\nu\}$ $(= W([I_{i_{r-1}}]))$ are canonical. Similarly we choose an element T_{r-2} of non-zero order in $[I_{i_{r-1}}]$ and a positive integer ν_{r-2} $(= w(T_{r-1}))$ in $\{0, \nu_{r-1} + \mathbb{N}\nu\}$, and we set

$$\begin{aligned} [I_{i_{r-2}}] &= k + T_{r-2}[I_{i_{r-1}}] \\ &= k + kT_{r-2} + T_{r-2}T_{r-1}k[T], \\ W([I_{i_{r-2}}]) &= \{0, \nu_{r-2}, \nu_{r-2} + \nu_{r-1} + \mathbb{N}\nu\}. \end{aligned}$$

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*In what follows $w\left(\sum_{i=\mu}^{\infty} \alpha_i t^i\right)$ denotes the order of the series $\sum_{i=\mu}^{\infty} \alpha_i t^i$ in t.

Thus we obtain a new canonical ring and also a canonical semigroup. Continuing in this manner we finally obtain the canonical ring

$$k + kT_1 + kT_1T_2 + \cdots + kT_1T_2\cdots T_{r-2} + k[T]T_1T_2\cdots T_{r-2}T_{r-1}$$

and the canonical semigroup *(p 263)*

$$\{0, \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \cdots + \nu_{r-1} + \mathbb{N}\nu\}$$

with

$$T_h \in kT_{h+1} + kT_{h+1}T_{h+2} + \cdots + k[T]T_{h+1}T_{h+2}\cdots T_{r-1},$$
$$(w(T_h) =)\ \nu_h \in \{\nu_{h+1}, \nu_{h+1} + \nu_{h+2}, \ldots, \nu_{h+1} + \nu_{h+2} + \cdots + \nu_{r-1} + \mathbb{N}\nu\}.$$

Section 4:

Given a ring H, the intersection of all canonical rings containing H is a canonical ring $^*H$ which we call the *canonical closure*[11] of H. Similarly $G = \{0, i_1, i_2, \ldots, i_{r-1} + \mathbb{N}\nu\}$ being a semigroup of non-negative integers ($\nu = (i_1, i_2, \ldots, i_{r-1} + \nu)$), the intersection of all canonical semigroups containing G is a canonical semigroup $^*G$; we call it the *canonical closure* of G.

It follows from this definition that $W(^*H)$ contains the canonical semigroup $^*W(H)$; but these two semigroups[12] are not necessarily equal, since $W(H)$ may be canonical without H being so.

Section 5:

Given a semigroup

$$G = \{0, i_1, i_2, \ldots, i_{r-1} + \mathbb{N}\nu\} \quad (\nu = (i_1, i_2, \ldots, i_{r-1}, i_{r-1} + \nu)),$$

the canonical closure $^*G$ of G is obtained as follows: We consider the semigroup $\{0, i_1 + G_1\}$ where G_1 is the semigroup of integers of the form

$$\alpha_2(i_2 - i_1) + \alpha_3(i_3 - i_1) + \cdots + \alpha_n(i_n - i_1),$$

where the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers. The semigroup $\{0, i_1 + G_1\}$ which then contains G is obviously contained in $^*G$. Note that the elements of G_1 which are less than $i_{h+1} - i_1$ are of the form

$$\alpha_2(i_2 - i_1) + \alpha_3(i_3 - i_1) + \cdots + \alpha_h(i_h - i_1);$$

the integers

$$\alpha_2(i_2 - i_1) + \alpha_3(i_3 - i_1) + \cdots + \alpha_n(i_n - i_1)$$

with $n \geq h+1,\ \alpha_n \neq 0$ are in fact greater than or equal to $i_{h+1} - i_1$. In particular the smallest element of G_1 is $i_2 - i_1$. Furthermore it follows that the elements of $\{0, i_1 + G_1\}$ which are less than i_{h+1} depend only on $i_1, i_2, \ldots, i_h$, and are linear combinations of these with integer coefficients. The semigroup $\{i_1 + G_1\}$ being contained in $^*G$, it is the same for $\{i_1 + {^*G_1}\}$ which contains $\{i_1 + G_1\} \supseteq G$, and is canonical. The construction of $^*G$ is thus reduced to the construction of the canonical closure of a semi-group of the form

$$G_1 = \{0, i'_1, i'_2, \ldots, i'_{r'-1} + \mathbb{N}\nu\};$$

for which we have $i'_{r'-1} \leq i_{r-1} - i_1$. The repetition of this construction reduces the proposed construction to that of the canonical closure of a semigroup G_N which itself reduces, for N sufficiently large, to the semigroup $\mathbb{N}\nu$ of all non-negative *(p 264)* multiples of ν. $\mathbb{N}\nu$ being its own canonical closure, the proposed procedure thus terminates. Note that the elements of $^*G$ which are thus constructed depend only on the elements of G which are not greater than themselves; and they are linear combinations of them with integer coefficients. Suppose in fact that this is proved for the closure $^*G_1$ of G_1. The elements of $^*G_1$ which are smaller than $i_{h+1} - i_h$ depend only on the elements of G_1 which are smaller than $i_{h+1} - i_h$, and they are their linear combinations with integer coefficients; now these latter ones in turn depend only on $i_1, i_2, \ldots, i_h$ and are their linear combinations with integer coefficients. It follows that the elements of $\{0, i_1 + {^*G_1}\} = {^*G}$ which are smaller than $i_{h+1} - i_h$ depend only on $i_1, i_2, \ldots, i_h$, and they are their linear combinations with integer coefficients.

Given a canonical semigroup

$$^*G = \{0, i_1, i_2, \ldots, i_{r-1} + \mathbb{N}\nu\} \quad (\nu = (i_1, i_2, \ldots, i_{r-1}, i_{r-1} + \nu)),$$

there exist only a finite number semigroups g such that $^*g = {^*G}$. In fact let

$$g = \{0, j_1, j_2, \ldots, j_s, j_{s+1}, \ldots\}$$

be such a semigroup. Let $j_1, j_2, \ldots, j_n$ of the integers $j_1, j_2, \ldots, j_s, \ldots$ be smaller than $i_{r+1} = i_{r-1} + 2\nu$. Since i_{r-1} and $i_{r-1} + \nu$ are linear combinations of $j_1, j_2, \ldots, j_n$ with integer coefficients, the $\gcd$ of these numbers is ν. Now to each system of positive integers smaller than $i_{r+1} = i_{r-1} + 2\nu$ whose $\gcd$ is ν, we can associate a multiple $j\nu$ of ν such that every semigroup of non-negative integers containing the system, contains all the multiples of ν larger than $j\nu$. Let $L\nu$ be the largest of the integers $j\nu$ which are thus associated to systems of positive multiples of ν smaller than $i_{r+1} = i_{r-1} + 2\nu$. The semigroups g for which we have $^*g = {}^*G$ contain then all the multiples of ν which are larger than $L\nu$ and they differ among themselves only by those elements which are smaller than $L\nu$.

Theorem 1. *The intersection of all the semigroups g such that $^*g = {}^*G$ is a semigroup g_χ such that $^*g_\chi = {}^*G$.*

Proof. Let g be a semigroup such that we have $^*g = {}^*G$ and that no proper sub semigroup of g has this property; we will show that $g = g_\chi$. Let i be the smallest element of g not in g_χ. Let $i_0 = 0, i_1, i_2, \dots, i_h$ be the elements of g and g_χ which are smaller than i. Since i is not contained in g_χ, the number i is not of the form

$$\alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_h i_h,$$

where $\alpha_1, \alpha_2, \dots, \alpha_h$ are non-negative integers. On the other hand g_χ being the intersection of all semigroups whose canonical closure is $^*G$, there exists a semigroup g' such that $^*g' = {}^*G$ and which does not contain the number i. Since the elements of $^*G = {}^*g$ which are smaller than i depend only on $i_1, i_2, \dots, i_h$, the *(p 265)*
semigroup g'' obtained by removing from g' all the the positive integers smaller than i except $i_1, i_2, \dots, i_h$ still has the property that $^*g'' = {}^*G$. It follows that the elements of $^*G$ which are smaller than or *equal* to i depend only on the numbers $i_1, i_2, \dots, i_h$; since g'' does not contain the number i. Therefore the canonical closure of the sub-semigroup of g obtained by removing from it the number i is still equal to $^*G$. This contradicts the choice of g. We then have $g_\chi = g$ and and consequently $^*g_\chi = {}^*G$. □

The semigroup g_χ defined in the statement of Theorem 1 is called the *characteristic sub-semigroup* of all the g such that $^*g = {}^*G$. It is clear that the semigroup g_χ is such that every proper sub-semigroup of g_χ has a canonical closure different than $^*g_\chi = {}^*G$. Conversely if g_χ is such that for every sub-semigroup g' of g_χ we have $^*g' \neq {}^*g_\chi$, then g_χ is its own characteristic sub-semigroup.

$g_\chi = \{0, i_1, i_2, \dots, i_{r-1}, i_r, \dots\}$ being the characteristic sub-semigroup of g, let us consider the integers $\chi_1, \chi_2, \dots, \chi_h$ defined in the following manner: $\chi_1 = i_1$; χ_2 is the smallest of the integers $i_1, i_2, \dots, i_r, \dots$ which is not of the form $\alpha_1\chi_1$ where α_1 is a non-negative integer; χ_3 is the smallest of the integers

$i_1, i_2, \ldots, i_r, \ldots$ which is not of the form $\alpha_1\chi_1 + \alpha_2\chi_2$ where α_1, α_2 are non-negative integers; finally $\chi_1, \chi_2, \ldots, \chi_n$ being defined, χ_{n+1} is the smallest of the integers $i_1, i_2, \ldots, i_r, \ldots$ which is not of the form

$$\alpha_1\chi_1 + \alpha_2\chi_2 + \cdots + \alpha_n\chi_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers. The numbers $\chi_1, \chi_2, \ldots, \chi_h$ defined in this manner are called *the characters* of g.

Theorem 2. *$\gamma_1 < \gamma_2 < \cdots < \gamma_\ell$ being a collection of positive integers, the set of characters of the semigroup g of integers of the form*

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 + \cdots + \alpha_\ell\gamma_\ell,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are non-negative integers, is contained in the collection $\gamma_1, \gamma_2, \ldots, \gamma_\ell$.

Proof. Let χ_i be the smallest of the characters $\chi_1, \chi_2, \ldots, \chi_h$ of g which is not contained in the set $\gamma_1, \gamma_2, \ldots, \gamma_\ell$. It follows from the definition of g_χ that χ_i is of the form $\alpha_1\gamma_1 + \alpha_2\gamma_2 + \cdots + \alpha_{\ell'}\gamma_{\ell'}$ where $\alpha_1, \alpha_2, \ldots, \alpha_{\ell'}$ are non-negative integers and where $\gamma_1, \gamma_2, \ldots, \gamma_{\ell'}$ are those integers among $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ which are smaller than χ_i. Since $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ are elements of the canonical closure of g_χ, every semigroup containing $\chi_1, \chi_2, \ldots, \chi_{i-1}$ contains also $\gamma_1, \gamma_2, \ldots, \gamma_{\ell'}$. This implies that the canonical closure of the semigroup of linear combinations with non-negative integer coefficients of $\chi_1, \chi_2, \ldots, \chi_{i-1}, \chi_{i+1}, \ldots, \chi_h$ contains χ_i, and it follows that g_χ is not a characteristic semigroup. Therefore the set $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ necessarily contains the set $\chi_1, \chi_2, \ldots, \chi_h$. □

Theorem 3. *g being the semigroup of linear combinations of [the integers][13] $0 <$ $\gamma_1 < \gamma_2 < \cdots < \gamma_\ell$ with non-negative integer coefficients, the integers* *(p 266)*

$$\nu_1, \nu_2, \ldots, \nu_{N-2}, \nu_{N-1}, \nu$$

with the property that

$${}^*g = \{0, \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \cdots + \nu_{N-1} + \mathbb{N}\nu\},$$

are obtained from $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ by the quasi-Jacobian algorithm of Du Val. The integers $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$ appear there as divisors, while the partial quotients*

*Du Val, loc. cit.

represent the number of times each divisor appears in the sequence $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$*. Conversely if the numbers*

$$\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$$

are obtained from $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ *by the quasi-Jacobian algorithm of Du Val, the partial quotients being the number of times each divisor appear in the sequence* $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$*, then the canonical closure of the semigroup of the integers of the form*

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 + \cdots + \alpha_\ell\gamma_\ell,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ *are non-negative integers, is* $^*g$*.*

Proof. ν being the greatest common divisor of the elements of g, we have $\gamma_1 \geq \nu$. If $\gamma_1 = \nu$, the semigroup g consists of the set of all multiples of $\gamma_1 = \nu$, and we have $g = {}^*g = \{\mathbb{N}\nu\}$. In this case the algorithm terminates at the first step. Let's assume that the proposition is proved for every set $\gamma_1', \gamma_2,', \ldots, \gamma_{\ell'}'$ for which $\gamma_1' < \gamma_1$, and prove it for for the set $\gamma_1, \gamma_2, \ldots, \gamma_\ell$. Let γ_i be the smallest of the integers $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ which is not divisible by γ_1. Let q be the quotient of γ_i by γ_1 and let us consider the semigroup Γ of linear combinations of $\gamma_i - q\gamma_1, \gamma_{i+1} - q\gamma_1, \ldots, \gamma_\ell - q\gamma_1, \gamma_1$ with non-negative integer coefficients. The semigroup $^*g$ clearly contains the semigroup $\{0, \gamma_1, 2\gamma_1, \ldots, q\gamma_1 + \Gamma\}$ which contains g. We then have

$$^*g = \{0, \gamma_1, 2\gamma_1, \ldots, q\gamma_1 + {}^*\Gamma\},$$
$$\nu_1 = \gamma_1,\ \nu_2 = \gamma_1, \ldots, \nu_q = \gamma_1$$

i.e.

$$^*\Gamma = \{0, \nu_{q+1}, \nu_{q+1} + \nu_{q+2}, \ldots, \nu_{q+1} + \cdots + \nu_{N-1} + \mathbb{N}\nu\}.$$

The integers $\gamma_i - q\gamma_1, \gamma_{i+1} - q\gamma_1, \ldots, \gamma_\ell - q\gamma_1, \gamma_1$ being the remainders of the $(i-1)$-st division of the algorithm applied to the numbers $\gamma_1, \gamma_2, \ldots, \gamma_\ell$, it suffices to show that the integers $\nu_{q+1}, \nu_{q+2}, \ldots, \nu_{N-1}, \nu$ are obtained by applying the algorithm to the integers $\gamma_i - q\gamma_1, \gamma_{i+1} - q\gamma_1, \ldots, \gamma_\ell - q\gamma_1, \gamma_1$. Now $\gamma_i - q\gamma_1$ being smaller than γ_1, this was assumed done. Conversely, if the numbers *(p 267)* $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$ are obtained from $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ by the quasi-Jacobian algorithm of Du Val, the canonical closure of the semigroup of linear combinations of $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ where coefficients are non-negative integers is

$$\{0, \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \cdots + \nu_{N-1} + \mathbb{N}\nu\},$$

which follows from the proposition we have just proved. □

$\chi_1, \chi_2, \ldots, \chi_h$ being the characters of g, the semigroup of linear combinations of $\chi_1, \chi_2, \ldots, \chi_h$ with coefficients being non-negative integers is the characteristic sub semigroup g_χ of g. It follows from theorems 3 and 2 that the integers $\nu_1, \nu_2, \ldots, \nu_{N-1}$ are obtained from the characters of g by the quasi-Jacobian algorithm of Du Val, and all the systems of non-negative integers $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ for which the algorithm produces the same result are obtained from the system of characters of g by adjoining to it numbers arbitrarily chosen from $^*g$.

Section 6:

Now let us consider a ring H and its canonical closure $^*H$. The ring H being of the form

$$H = k + kS_{i_1} + kS_{i_2} + \cdots + k[T]S_{i_h}$$

its canonical closure $^*H$ can be constructed as follows: Denote by H_1 the ring

$$[I_{i_1}] = \sum k \left(\frac{S_{i_2}}{S_{i_1}}\right)^{\alpha_2} \left(\frac{S_{i_3}}{S_{i_1}}\right)^{\alpha_3} \cdots \left(\frac{S_{i_{h-1}}}{S_{i_1}}\right)^{\alpha_{h-1}} + k[T]\frac{S_{i_h}}{S_{i_1}},$$

where the summation is over all exponent systems of non-negative integers $\alpha_2, \alpha_3, \ldots, \alpha_{h-1}$ such that $\alpha_2(i_2 - i_1) + \alpha_3(i_3 - i_1) + \cdots + \alpha_{h-1}(i_{h-1} - i_1)$ is less than $i_h - i_1$. The canonical closure $^*H$ of H clearly contains

$$k + H_1 S_{i_1}$$

which contains H and we have $^*H = k + {}^*H_1 S_{i_1}$, where we denoted by $^*H_1$ the canonical closure of H_1. In general, H_i being defined, denote by H_{i+1} the ring obtained from H_i in the same way H_1 is obtained from H. It is clear that for N sufficiently large, H_N is isomorphic to $K[T]$. Let T_{i+1} be an element of positive order in H_i. Then we obviously have

$$\begin{aligned} {}^*H &= k + kT_1 + {}^*H_2 T_1 T_2 \qquad \text{(with } T = S_i\text{)} \\ &= k + kT_1 + kT_1T_2 + {}^*H_3 T_1 T_2 T_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &= k + kT_1 + kT_1T_2 + \cdots + kT_1T_2\cdots T_{N-1} + {}^*H_N T_1 T_2 \cdots T_{N-1} T_N \\ &= k + kT_1 + kT_1T_2 + \cdots + kT_1T_2\cdots T_{N-1} + k[T] T_1 T_2 \cdots T_{N-1} T_N. \end{aligned}$$

Remark. For any integer n, the ring $k + H_1 S_{i_1} \mod t^n$ depends only on $H \mod t^n$. To prove this it suffices to show that $H_1 \mod t^{n-i_1}$ depends only on $H \mod t^n$. Similarly the ring $k + H_2 T_2 \mod t^{n-i_2}$ depends only on $H_1 \mod t^{n-i_1}$. The ring $k + kT_1 + H_2 T_1 T_2 \mod t^n$ then depends only on $H \mod t^n$. Continuing in this manner we obtain eventually that (p 268)

$$ {}^*H = k + kT_1 + kT_1T_2 + \cdots + kT_1T_2\cdots T_{N-1} + H_N T_1 T_2 \cdots T_N \mod t^n $$

depends only on $H \mod t^n$.

Lemma 5. *If $H \mod t^n$ is equal to ${}^*H \mod t^n$, then the set ${}^*H \mod t^{n+1}$ is equal to one of the sets*

$$ k + kS_{i_1} + kS_{i_2} + \cdots + kS_{i_{\ell-1}} + [I_{i_\ell}]S_{i_\ell} \mod t^{n+1} \quad (i_\ell < n+1). $$

Proof. The set ${}^*H \mod t^n$ being the same as $H \mod t^n$, the set ${}^*H \mod t^{n+1}$, which contains the set $H \mod t^{n+1}$, consists of the elements of the form

$$ S + \alpha {}^*S_n \mod t^{n+1} $$

where S is an element of H, ${}^*S_n$ a fixed element of order n belonging to ${}^*H$, and α an element of k. Hence every ring $H' \mod t^{n+1}$, contained in the ring $H \mod t^{n+1}$ is identical to $H \mod t^{n+1}$, if it is contained in ${}^*H \mod t^{n+1}$ without it being identical. Consider now the ring

$$ k + S_{i_1}[I_{i_1}] \mod t^{n+1} $$

which contains $H \mod t^{n+1}$ and which is contained in ${}^*H \mod t^{n+1}$. After what we have just noted, the ring $k + S_{i_1}[I_{i_1}] \mod t^{n+1}$ is identical to one of the two rings

$$ {}^*H \mod t^{n+1}, \quad H \mod t^{n+1}. $$

If it is not identical to the first, we have $[I_{i_1}] = I_{i_1}/S_{i_1} \mod t^{n+1-i_1}$. As ${}^*[I_{i_1}] \mod t^{n+1-i_1}$ depends only on $[I_{i_1}] \mod t^{n+1-i_1}$, the sets

$$ {}^*[I_{i_1}] \mod t^{n+1-i_1}, \quad k + \frac{S_{i_2}}{S_{i_1}} {}^*[I_{i_2}] \mod t^{n+1-i_1} $$

will be identical, since I_{i_2}/S_{i_1} is the set of elements of positive order in I_{i_1}/S_{i_1}. It follows that ${}^*H \mod t^{n+1}$ is identical to one of the rings

$$ k + S_{i_1}[I_{i_1}] \mod t^{n+1}, \quad k + kS_{i_1} + {}^*[I_{i_2}]S_{i_2} \mod t^{n+1}. $$

If ${}^*H \mod t^{n+1}$ is neither identical to $k + S_{i_1}[I_{i_1}] \mod t^{n+1}$ nor to

$$k + kS_{i_1} + {}^*[I_{i_2}]S_{i_2} \mod t^{n+1}$$

these two rings are identical to $H \mod t^{n+1}$. Under these conditions we have $[I_{i_2}] \equiv I_{i_2}/S_{i_2} \mod t^{n+1-i_2}$, from which we can conclude the identity of the two sets

$$ {}^*[I_{i_2}] \mod t^{n+1-i_2}, \quad k + {}^*[I_{i_2}]\frac{S_{i_2}}{S_{i_1}} \mod t^{n+1-i_2}.$$

${}^*H \mod t^{n+1}$ is then identical to one of the sets *(p 269)*

$$k + [I_{i_1}]S_{i_1} \mod t^{n+1}, \quad k + kS_{i_1} + [I_{i_2}]S_{i_2} \mod t^{n+1},$$
$$k + kS_{i_1} + kS_{i_2} + {}^*[I_{i_3}]S_{i_3} \mod t^{n+1}.$$

Continuing in this manner we can show that ${}^*H \mod t^{n+1}$ is identical to one of the sets

$$\begin{array}{ll}
k + [I_{i_1}]S_{i_1} & \mod t^{n+1} \\
k + kS_{i_1} + [I_{i_2}]S_{i_2} & \mod t^{n+1} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
k + kS_{i_1} + kS_{i_2} + \cdots + [I_{i_\ell}]S_{i_\ell} & \mod t^{n+1} \\
k + kS_{i_1} + kS_{i_2} + \cdots + kS_{i_\ell} + {}^*[I_{i_{\ell+1}}]S_{i_{\ell+1}} & \mod t^{n+1}.
\end{array}$$

Now for $i_{\ell+1} \geq n+1$, the last one of these sets is $H \mod t^{n+1}$. Then ${}^*H \mod t^{n+1}$ is identical to one of the sets

$$k + kS_{i_1} + kS_{i_2} + \cdots + [I_{i_\ell}]S_{i_\ell} \mod t^{n+1}$$

for $i_\ell \leq n$. □

$X_1, X_2, \ldots, X_n$ being power series in t with positive orders, we denote by $k[X_1, X_2, \ldots, X_n]$ the ring formed by the series of the form

$$\sum \alpha_{j_1 j_2 \cdots j_n} X_1^{j_1} X_2^{j_2} \cdots X_n^{j_n}$$

where $\alpha_{j_1 j_2 \cdots j_n} \in k$ and the summation is over all systems of non-negative integers $j_1, j_2, \ldots j_n$.

Lemma 6. *The elements $Y_1, Y_2, \ldots, Y_\nu$ of $^*H$ being chosen such that $w(Y_j)$ is the smallest element of $W(^*H)$ which is not contained in $W(k[Y_1, Y_2, \ldots, Y_{j-1}])$, if the elements $Y'_1, Y'_2, \ldots, Y'_{\nu-1}$ are respectively congruent to $Y_1, Y_2, \ldots, Y_{\nu-1}$ $\bmod t^{w(Y_\nu)}$, then the smallest element of $W(^*H)$ not contained in $W(k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}])$ is $w(Y_\nu)$.*

Proof. The rings

$$^*H \mod t^{w(Y_\nu)},\ k[Y_1, Y_2, \ldots, Y_{\nu-1}] \mod t^{w(Y_\nu)},\ k[Y'_1, \ldots, Y'_{\nu-1}] \mod t^{w(Y_\nu)}$$

being clearly identical, it suffices to show that $k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}]$ does not contain an element of order $w(Y_\nu)$. Every element of $k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}] \mod t^{w(Y_\nu)+1}$ being of the form

$$P(Y'_1, Y'_2, \ldots, Y'_{\nu-1}) \mod t^{w(Y_\nu)+1}$$

where $P(Y'_1, Y'_2, \ldots, Y'_{\nu-1})$ is a polynomial with coefficients in k, it suffices to show that $w(P(Y'_1, Y'_2, \ldots, Y'_{\nu-1}))$ cannot be equal to $w(Y_\nu)$. If the polynomial $P(Y'_1, Y'_2, \ldots, Y'_{\nu-1})$ contains a [nonzero][14] constant term, then $Y'_1, Y'_2, \ldots, Y'_{\nu-1}$ being elements of positive order we have $w(P(Y'_1, Y'_2, \ldots, Y'_{\nu-1})) = 0 \neq w(Y_\nu)$. If

$$P(Y'_1, Y'_2, \ldots, Y'_{\nu-1})$$

contains terms of degree 1 without containing a [nonzero] constant term, then we can write it in the form (p 270)

$$P_1(Y'_1, Y'_2, \ldots, Y'_{j-1}) + \beta Y'_j + P_2(Y'_1, Y'_2, \ldots, Y'_{\nu-1})$$

with $\beta \neq 0$; $P_2(Y'_1, Y'_2, \ldots, Y'_{\nu-1})$ being the sum of terms of positive degree with respect to the set $Y'_j, Y'_{j+1}, \ldots, Y'_{\nu-1}$ except the term $\beta Y'_j$. $w(P_2(Y'_1, Y'_2, \ldots, Y'_{\nu-1}))$ is then greater than $w(Y'_j)$ which is by definition different than the order of

$$P_1(Y'_1, Y'_2, \ldots, Y'_{j-1}) \equiv P_1(Y_1, Y_2, \ldots, Y_{j-1}) \mod t^{w(Y_\nu)}.$$

We then have

$$w(P(Y'_1, Y'_2, \ldots, Y'_{\nu-1})) = \min(w(Y'_j)), \quad w(P_1(Y'_1, Y'_2, \ldots, Y'_{j-1})) < w(Y_\nu).$$

Finally if $P(Y'_1, Y'_2, \ldots, Y'_{\nu-1})$ contains neither a term of degree 1 nor of degree 0, then we can write

$$P(Y'_1, Y'_2, \ldots, Y'_{\nu-1}) \equiv P(Y_1, Y_2, \ldots, Y_{\nu-1}) \mod t^{w(Y_\nu)+1}.$$

$w(P(Y_1, Y_2, \dots, Y_{\nu-1}))$ being different than $w(Y_\nu)$, it is the same for

$$w(P(Y'_1, Y'_2, \dots, Y'_{\nu-1})).$$

□

Lemma 7. *$Y_1, Y_2, \dots, Y_{\nu-1}, Y_\nu$ and $Y'_1, Y'_2, \dots, Y'_{\nu-1}$ having the same properties as in the statement of Lemma 6, if the canonical closure of $k[Y_1, Y_2, \dots, Y_{\nu-1}]$ does not contain an element of order $w(Y_\nu)$, then it is the same for the canonical closure of $k[Y'_1, Y'_2, \dots, Y'_{\nu-1}]$.*

Proof. Let $i_0 = 0, i_1, i_2, \dots, i_\mu, \dots$ be the orders of the elements of $k[Y'_1, Y'_2, \dots, Y'_{\nu-1}]$ written in increasing order and let I'_{i_μ} be the set of elements of $k[Y'_1, Y'_2, \dots, Y'_{\nu-1}]$ whose orders are not smaller than i_μ. Denote by S'_{i_ℓ} an element of order i_ℓ of $k[Y'_1, Y'_2, \dots, Y'_{\nu-1}]$, and by $\mathcal{H}'$ [15] the canonical closure of $k[Y'_1, Y'_2, \dots, Y'_{\nu-1}]$. The rings

$$^*H \mod t^{w(Y_\nu)}, \quad \mathcal{H}' \mod t^{w(Y_\nu)}, \quad k[Y'_1, Y'_2, \dots, Y'_{\nu-1}] \mod t^{w(Y_\nu)}$$

being identical, it follows from Lemma 5 that the ring $\mathcal{H}' \mod t^{w(Y_\nu)+1}$ is identical to one of the rings

$$k + kS'_{i_1} + kS'_{i_2} + \dots + [I'_{i_\ell}]S'_{i_\ell} \mod t^{w(Y_\nu)+1}$$

with $i_\ell < w(Y_\nu)$. Let μ be the smallest of these integers ℓ for which this identity holds. If $\mu = 0$, then $\mathcal{H}' \mod t^{w(Y_\nu)+1}$ is identical to $k[Y'_1, Y'_2, \dots, Y'_{\nu-1}] \mod t^{w(Y_\nu)+1}$ which does not contain an element of order $w(Y_\nu)$. Suppose then that μ is positive. To show that $\mathcal{H}'$ does not contain an element of order $w(Y_\nu)$, it suffices to show that $[I'_{i_\mu}]$ does not contain an element of order $w(Y_\nu) - i_\mu$. Let I_{i_μ} and $^*I_{i_\mu}$ be the sets of elements of order not smaller than i_μ of $k[Y_1, \dots, Y_{\nu-1}]$ and $^*H$. The rings

$$^*H \mod t^{w(Y_\nu)}, \ k[Y_1, Y_2, \dots, Y_{\nu-1}] \mod t^{w(Y_\nu)}, \ k[Y'_1, Y'_2, \dots, Y'_{\nu-1}] \mod t^{w(Y_\nu)}$$

being identical, it is the same for the sets *(p 271)*

$$[^*I_{i_\mu}] \mod t^{w(Y_\nu) - i_\mu}$$
$$I_{i_\mu}/S_{i_\mu} \mod t^{w(Y_\nu) - i_\mu}, \quad I'_{i_\mu}/S'_{i_\mu} \mod t^{w(Y_\nu) - i_\mu},$$

where S_{i_μ} is an element of $k[Y_1, Y_2, \ldots, Y_{\nu-1}]$, such that we have

$$S_{i_\mu} \equiv S'_{i_\mu} \mod t^{w(Y_\nu)}.$$

It follows that we can associate to every element Z' of I'_{i_μ}/S'_{i_μ} an element Z of I_{i_μ}/S_{i_μ} in such a way that we have

$$Z = Z' \mod t^{w(Y_\nu)-i_\mu}.$$

Let us consider in particular a set of elements $Z'_1, Z'_2, \ldots, Z'_\rho$ of I'_{i_μ}/S'_{i_μ} chosen in the following way:

(1) Z'_1 is an element of smallest positive order in I'_{i_μ}/S'_{i_μ},

(2) $Z'_1, Z'_2, \ldots, Z'_{j-1}$ being chosen, we choose Z'_j in such a way that $w(Z'_j)$ is the smallest positive element of $W(I'_{i_\mu}/S'_{i_\mu})$ which is not contained in $W(k[Z'_1, Z'_2, \ldots, Z'_{j-1}])$,

(3) $w(Z'_\rho) < w(Y_\nu) - i_\mu + 1$ and every element of $W(I'_{i_\mu}/S'_{i_\mu})$ smaller than $w(Y_\nu) - i_\mu + 1$ is contained in $W(k[Z'_1, Z'_2, \ldots, Z'_\rho])$.

$k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}] \mod t^{w(Y_\nu)+1}$ being distinct than $\mathcal{H}' \mod t^{w(Y_\nu)+1}$ while $k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}] \mod t^{w(Y_\nu)+1}$ is identical to $\mathcal{H}' \mod t^{w(Y_\nu)}$, the ring $k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}]$ cannot contain elements of orders $w(Y_\nu)$. It follows that the numbers $w(Z'_1), w(Z'_2), \ldots, w(Z'_\rho)$ are smaller than $w(Y_\nu) - i_\mu$.The conditions imposed on the choice of $Z'_1, Z'_2, \ldots, Z'_\rho$ implies further the identity of the rings

$$[I'_{i_\mu}] \mod t^{w(Y_\nu)-i_\mu+1}, \quad k[Z'_1, Z'_2, \ldots, Z'_\rho] \mod t^{w(Y_\nu)-i_\mu+1};$$

It suffices then to show that $k[Z'_1, Z'_2, \ldots, Z'_\rho]$ does not contain an element of order $w(Y_\nu) - i_\mu$. Now let $Z_1, Z_2, \ldots, Z_\rho$ be elements of I_{i_μ}/S_{i_μ} such that we have

$$Z_j \equiv Z'_j \mod t^{w(Y_\nu)-i_\mu} \quad (j = 1, 2, \ldots, \rho).$$

The canonical closure of $k[Y_1, Y_2, \ldots, Y_{\nu-1}]$ not containing any element of order $w(Y_\nu)$, the ring $k[Z_1, Z_2, \ldots, Z_\rho]$ does not contain any element of order $w(Y_\nu) - i_\mu$. The elements $Z_1, Z_2, \ldots, Z_\rho, Z_{\rho+1} = Y_\nu/S_{i_\mu}$ of $[{}^*I_{i_\mu}]$ and $Z'_1, Z'_2, \ldots, Z'_\rho$ fulfill the conditions of the statement of Lemma 6 with respect to the canonical ring $[{}^*I_{i_\mu}]$. The ring $k[Z'_1, Z'_2, \ldots, Z'_\rho]$ then cannot contain elements of order $w(Z_{\rho+1}) = w(Y_\nu) - i_\mu$. □

Let us now consider a set of elements $X_1, X_2, \ldots, X_m$ of ${}^*H$ chosen as follows: X_1 is an element of smallest positive order in ${}^*H$; $X_1, X_2, \ldots, X_{\ell-1}$ being chosen, X_ℓ is an element of ${}^*H$ such that $w(X_\ell)$ is the smallest element of

$W({}^*H)$ which is not contained in $W(\mathcal{H}_{\ell-1})$, where $\mathcal{H}_{\ell-1}$[16] is the canonical closure of $k[X_1, X_2, \ldots, X_{\ell-1}]$. The elements of $W({}^*H)$ being linear combinations with non-negative integer coefficients of some finite number of elements, the elements $X_1, X_2, \ldots, X_\ell, \ldots$ chosen in this manner can only be finite. A set of such elements $(X_1, X_2, \ldots, X_m)$ will be called in what follows a *base* of ${}^*H$. *(p 272)*

Theorem 4. *$(X_1, X_2, \ldots, X_m)$ being a base of ${}^*H$, the integers*

$$w(X_1), w(X_2), \ldots, w(X_m)$$

depend on H and they constitute a subset of the characters of H.

Let us first prove the following proposition which will facilitate the proof of this theorem.

Lemma 8. *$\mathcal{H}_\ell$ being the canonical closure of $k[X_1, X_2, \ldots, X_\ell]$ where $X_1, X_2, \ldots, X_m$ is a base of ${}^*H$, one can choose the elements $Y_1, Y_2, \ldots, Y_\nu, \ldots$ of $\mathcal{H}_\ell$ satisfying the conditions of the statement of Lemma 6 considered for the ring $\mathcal{H}_\ell$ (i.e. $w(Y_j)$ is the smallest element of $w(\mathcal{H}_\ell)$ not contained in $W(k[Y_1, Y_2, \ldots, Y_{j-1}])$) in such a manner that the sequence $Y_1, Y_2, \ldots, Y_\nu, \ldots$ contains the set $X_1, X_2, \ldots, X_\ell$.*

Proof. For $\ell = 1$, we clearly have $\mathcal{H}_1 = k[X_1]$ and we can set $Y_1 = X_1$. Assume that the proposition is proved for ℓ and let us prove it for $\ell + 1$. Let $Y_1, Y_2, \ldots, Y_\nu$ be the elements chosen from $\mathcal{H}_\ell$ whose orders are smaller than $w(X_{\ell+1})$. The elements of $W(\mathcal{H}_\ell)$ which are smaller than $w(X_{\ell+1})$ are then the same as those of $W(k[Y_1, Y_2, \ldots, Y_\nu])$. The smallest element of $W(\mathcal{H}_{\ell+1})$ not contained in $W(\mathcal{H}_\ell)$ being $w(X_{\ell+1})$, set $Y_{\nu+1} = X_{\ell+1}$, and choose $Y_{\nu+2}, Y_{\nu+3}, \ldots$ from $\mathcal{H}_{\ell+1}$ in accordance with the statement of Lemma 6 with respect to $\mathcal{H}_{\ell+1}$. The sequence

$$Y_1, Y_2, \ldots, Y_\nu, Y_{\nu+1}, \ldots$$

then satisfies for $\mathcal{H}_{\ell+1}$ the conditions of the statement of the proposition which we wanted to prove. □

Proof of Theorem 4. Let $X_1, X_2, \ldots, X_m$ and $X'_1, X'_2, \ldots, X'_{m'}$ be two bases of ${}^*H$. If the integers $w(X_1), w(X_2), \ldots, w(X_m)$ and the integers $w(X'_1), w(X'_2), \ldots, w(X'_m)$ are not the same, then at least one of the integers $(w(X_1), w(X_2), \ldots, w(X_m), w(X'_1), w(X'_2), \ldots, w(X'_m))$ does not belong to one of the sets $(w(X_1),$

$w(X_2), \ldots, w(X_m))$, $(w(X'_1), w(X'_2), \ldots, w(X'_m))$. Let $w(X'_{\ell+1})$ be the smallest of these integers which do not belong to one of these sets, and consider the canonical closures $\mathcal{H}_\ell$, $\mathcal{H}'_\ell$ of the rings $k[X_1, X_2, \ldots, X_\ell]$, $k[X'_1, X'_2, \ldots, X'_\ell]$. Because of the way X'_j, X_j are chosen, it follows that the rings $\mathcal{H}_\ell \mod t^{w(X_{\ell+1})}$, $\mathcal{H}'_\ell \mod t^{w(X'_{\ell+1})}$ are respectively identical to the rings ${}^*H \mod t^{w(X_{\ell+1})}$, ${}^*H \mod t^{w(X'_{\ell+1})}$. $w(X_{\ell+1})$ being by definition larger than $w(X'_{\ell+1})$, the ring $\mathcal{H}_\ell$ must contain an element of order $w(X'_{\ell+1})$. Now let $(Y_1, Y_2, \ldots, Y_\nu, \ldots)$ be a set of elements of $\mathcal{H}_\ell$ chosen in accordance with the statement of Lemma 8 and *(p 273)* let $Y_1, Y_2, \ldots, Y_\nu$ be those elements of this set whose orders are smaller than $w(X'_{\ell+1})$. The rings

$$ {}^*H \mod t^{w(X'_{\ell+1})}, \quad \mathcal{H}_\ell \mod t^{w(X_{\ell+1})}, \quad \mathcal{H}'_\ell \mod t^{w(X'_{\ell+1})} $$
$$ k[Y_1, Y_2, \ldots, Y_\nu] \mod t^{w(X'_{\ell+1})} $$

being identical, there exist elements $Y'_1, Y'_2, \ldots, Y'_\nu$ of $\mathcal{H}'_\ell$ such that

$$ Y'_j = Y_j \mod t^{w(X'_{\ell+1})} \quad (j = 1, 2, \ldots, \nu). $$

The canonical closure of $k[Y'_1, Y'_2, \ldots, Y'_\nu]$ which is contained in $\mathcal{H}'_\ell$ cannot contain any element of order $w(X'_{\ell+1})$. Therefore the canonical closure of $k[Y_1, Y_2, \ldots, Y_\nu]$ which is none other than $\mathcal{H}_\ell$ (since the set $(Y_1, Y_2, \ldots, Y_\nu)$ contains the set $(X_1, X_2, \ldots, X_\ell)$) does not contain an element of order $w(X'_{\ell+1})$ (Lemma 7). Therefore $w(X_{\ell+1})$ is equal to $w(X'_{\ell+1})$ which contradicts the hypothesis.

That the numbers $w(X_1), w(X_2), \ldots, w(X_n)$ constitutes a subset of the characters of ${}^*H$ is established as follows: $w(X_1)$ being the smallest element of $W({}^*H)$ we have $w(X_1) = \chi_1$. Assume that $w(X_\ell)$ is the smallest of the numbers $w(X_1), w(X_2), \ldots, w(X_n)$[17] which is not a character of ${}^*H$. $w(X_\ell)$ would then be contained in the canonical closure of the semigroup generated by the elements of $W({}^*H)$ which are smaller than $w(X_\ell)$. Now the elements of $W({}^*H)$ which are smaller than $w(X_\ell)$ are contained in $W(\mathcal{H}_{\ell-1})$. We then have $w(X_\ell) \in W(\mathcal{H}_{\ell-1})$ which contradicts the choices of the X_j. □

In what follows we will call the numbers

$$ w(X_1) = {}^*\chi_1, w(X_2) = {}^*\chi_2, \ldots, w(X_m) = {}^*\chi_m $$

the base characters of ${}^*H$. It follows immediately from the definition of a base of ${}^*H$ and from Theorem 4 that every system of elements ${}^*X_1, {}^*X_2, \ldots, {}^*X_m$ of ${}^*H$

such that $w({}^*X_1) = {}^*\chi_1, w({}^*X_2) = {}^*\chi_2, \ldots, w({}^*X_m) = {}^*\chi_m$ constitutes a base of ${}^*H$.

A set of elements $Y_1, Y_2, \ldots, Y_\nu$ of H is called a *system of generators*, if ${}^*H$ is the canonical closure of $k[Y_1 - \eta_1, Y_2 - \eta_2, \ldots, Y_\nu - \eta_\nu]$ where $\eta_1, \eta_2, \ldots, \eta_\nu$ denote the constant terms of $Y_1, Y_2, \ldots, Y_\nu$.

$X_1, X_2, \ldots, X_m$being a base of ${}^*H$, let us consider a set of elements $Y_1, Y_2, \ldots, Y_m$ chosen in the following manner:

$$\begin{array}{ll} Y_1 = X_1 + X_1' & X_1' \in k \\ Y_2 = X_2 + X_2' & X_2' \in \mathcal{H}_1 \\ \cdots\cdots\cdots\cdots & \cdots\cdots \\ Y_m = X_m + X_m' & X_m' \in \mathcal{H}_{m-1} \end{array}$$

where $\mathcal{H}_i$ denotes the canonical closure of $k[X_1, X_2, \ldots, X_i]$; the elements $Y_1, Y_2,$ *(p 274)* $\ldots, Y_m$ clearly constitutes a system of generators for ${}^*H$. Conversely every system of generators contains a subset chosen in this manner. In fact $Y_1, Y_2, \ldots, Y_\nu$ being a system of generators for ${}^*H$, denote by $\eta_1, \eta_2, \ldots, \eta_\nu$ the constant terms of $Y_1, Y_2, \ldots, Y_\nu$. At least one of the integers $w(Y_1 - \eta_1), w(Y_2 - \eta_2), \ldots, w(Y_\nu - \eta_\nu)$ is then equal to ${}^*\chi_1$, let's say $w(Y_1 - \eta_1) = {}^*\chi_1$. We can then set $X_1 = Y_1 - \eta_1$. Since $W(\mathcal{H}_1)$ contains all the elements of $W({}^*H)$ which are smaller than ${}^*\chi_2$, we can choose $X_i' \in \mathcal{H}_1$ in such a way that we have

$$w(Y_i - X_i') \geq {}^*\chi_2 \quad (i = 2, 3, \ldots, \nu).$$

At least one of the integers $w(Y_i - X_i')$ is equal to ${}^*\chi_2$; otherwise the canonical closure of $k[X_1, Y_2 - X_2', \ldots, Y_\nu - X_\nu']$ which is by definition is identical to ${}^*H$ does not contain any element of order ${}^*\chi_2$. Let $w(Y_2 - X_2') = {}^*\chi_2$. We can then set $X_2 = Y_2 - X_2'$ and so on. It follows from these considerations that every system of generators of ${}^*H$ contains at least m elements, m being the number of the base characters of ${}^*H$; we call this number *the dimension of* ${}^*H$.

Section 7: ${}^*H = k + kT_1 + kT_1T_2 + \cdots + k[T]T_1T_2\cdots T_{N-1}$ being a canonical ring, the characters, as well as the base characters, for the rings

$$[I_{i_h}] = {}^*H_h = k + kT_{h+1} + \cdots + k[T]T_{h+1}T_{h+2}\cdots T_{N-1}$$

are invariants of ${}^*H$. The characters of ${}^*H_h$ are clearly determined by those of ${}^*H$. But it is not so for the base characters of ${}^*H_h$.

Consider for example the rings

$$\left.\begin{array}{l} {}^*H = k + kt^{4\nu}(1+t) + kt^{6\nu}(1+t) + kt^{7\nu}(1+t) + k[t]t^{8\nu}, \\ {}^*H' = k + kt^{4\nu} + kt^{6\nu}(1+t) + kt^{7\nu}(1+t) + k[t]t^{8\nu}. \end{array}\right\} \quad (\nu > 1)$$

It can easily be checked that these two rings are canonical and that their characters which are also those of the semigroup

$$W({}^*H) = W({}^*H') = \{0, 4\nu, 6\nu, 7\nu, 8\nu+1, 8\nu+2, 8\nu+3, \dots\}$$

are the same. These characters are clearly $4\nu, 6\nu, 7\nu, 8\nu+1$. Let us now construct a base for ${}^*H$: We can clearly set $X_1 = t^{4\nu}(1+t)$; $k[X_1]$ is a canonical ring and the smallest element of $W({}^*H)$ not contained in $W(k[X_1])$ is 6ν; we can then set $X_2 = t^{6\nu}(1+t)$. The canonical closure of $k[X_1, X_2]$ is

$$\overline{k[X_1, X_2]} = k + kt^{4\nu}(1+t) + kt^{6\nu}(1+t) + k[t]t^{8\nu}.$$

We can then choose $X_3 = t^{7\nu}(1+t)$ as the third element of the base of ${}^*H$. The canonical closure of $k[X_1, X_2, X_3]$ then being equal to ${}^*H$, the base characters of (p 275) ${}^*H$ are $4\nu, 6\nu, 7\nu$. In a similar manner, we observe that the elements $X'_1 = t^{4\nu}$, $X'_2 = t^{6\nu}(1+t)$, $X'_3 = t^{7\nu}(1+t)$ constitutes a base for ${}^*H'$. The base characters of ${}^*H$ and ${}^*H'$ are then the same. Let us now calculate the base characters of the rings

$$\begin{array}{l} {}^*H_1 = k + kt^{2\nu} + kt^{3\nu} + k[t]t^{4\nu}, \\ {}^*H'_1 = k + kt^{2\nu}(1+t) + kt^{3\nu}(1+t) + k[t]t^{4\nu}. \end{array}$$

A base of ${}^*H_1$[18] is formed by $t^{2\nu}, t^{3\nu}, t^{4\nu+1}$, while the elements $t^{2\nu}(1+t), t^{3\nu}(1+t)$ form a base of ${}^*H'_1$, since the canonical closure of $k[t^{2\nu}(1+t), t^{3\nu}(1+t)]$ contains the element

$$t^{4\nu}(1+t)^2 - t^{2\nu}(1+t)\left(\frac{t^{3\nu}(1+t)}{t^{2\nu}(1+t)}\right)^2 = t^{4\nu+1}(1+t)$$

whose order is $4\nu+1$. The base characters of ${}^*H_1$ are then $2\nu, 3\nu, 4\nu+1$ while those of ${}^*H'_1$ are $2\nu, 3\nu$.

The base characters of the rings $[I_{i_h}] = {}^*H_h$ *constitute then new invariant elements for* ${}^*H$.

The following considerations allow us to determine successively the base characters of the ${}^*H_h$. *Consider an arbitrary element of positive order in* ${}^*H$. *Let* T *be* *this element and let* $(X_1, X_2, \ldots, X_m)$ *be a base of* ${}^*H$. *Denote by* ${}^*\chi_i$ *the smallest of the numbers*

$${}^*\chi_1 = w(X_1), {}^*\chi_2 = w(X_2), \ldots, {}^*\chi_m = w(X_m), {}^*\chi_{m+1} = \infty^{19}$$

such that the canonical closure of $k[X_1, X_2, \ldots, X_{i-1}, T]$ *contains*[20] *an element of order* ${}^*\chi_i$. *The elements* $T, TX_1, TX_2, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_m$ *constitute then a base of* $k + {}^*HT$ *which is canonical.* In fact

$$\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]}$$

being the canonical closure of $k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]$, the canonical closure of $k[T, TX_1, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_m]$ clearly contains the ring

$$k + T\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]}.$$

As $\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]}$ contains an element of order ${}^*\chi_i$, we have

$$\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]} = {}^*H.$$

The canonical closure of $k[T, TX_1, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_m]$ is then identical to

$$k + T\,{}^*H$$

which it contains; since the ring $k[T, TX_1, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_m]$ is itself contained in $k + T\,{}^*H$. Then to show that the elements

$$T, TX_1, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_m$$

constitute a base of $k + T\,{}^*H$, it suffices to show that the canonical closures of the rings *(p 276)*

$$\begin{array}{ll} k[T, TX_1, \ldots, TX_j] & (1 \leq j < i-1) \\ k[T, TX_1, \ldots, TX_{i-1}] & \\ k[T, TX_1, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_h] & (n > h \geq i+1) \end{array}$$

do not contain elements of orders, respectively,

$$w(TX_{j+1}), w(TX_{i+1}), w(TX_{h+1}).$$

Now these closures are identical respectively to

$$k + T\overline{k[X_1, X_2, \ldots, X_j, T]}$$
$$k + T\overline{k[X_1, X_2, \ldots, X_{i-1}, T]}$$
$$k + T\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_h, T]},$$

where the overline symbol denote always the canonical closure of the corresponding ring. It then suffices to show that the canonical closures of the rings $k[X_1, \ldots, X_j, T], k[X_1, \ldots, X_{i-1}, T], k[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_h, T]$ do not contain elements of orders $w(X_{j+1}), w(X_{i+1}), w(X_{h+1})$, respectively. Now the fact that the canonical closure of $k[X_1, \ldots, X_j, T]$ for $j < i - 1$ does not contain any element of order $w(X_{j+1})$ follows from the definition of i. If the ring

$$k[X_1, \ldots, X_{i-1}, T]$$

contains an element of order $w(X_{i+1})$ or the ring

$$k[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_h, T]$$

an element of order $w(X_{h+1})$, the canonical closure of one of the rings

$$\begin{aligned} &k[X_1, X_2, \ldots, X_{i-1}, X_{i+2}, \ldots, X_m, T], \\ &k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_h, X_{h+2}, \ldots, X_m, T], \qquad \text{for } h < m - 1, \\ &k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{m-1}, T], \qquad \text{for } h = m - 1, \end{aligned}$$

contains a system of elements of orders ${}^*\chi_1, {}^*\chi_2, \ldots, {}^*\chi_m$ and as a consequence a base of ${}^*H$. This implies the existence of a system of generators of ${}^*H$ containing only $m - 1$ elements, contrary to what has been established above (see Section 6).

The base characters of $k + T\,{}^*H$ are then

$$w(T), w(T)+{}^*\chi_1, w(T)+{}^*\chi_2, \ldots, w(T)+{}^*\chi_{i-1}, w(T)+{}^*\chi_{i+1}, \ldots, w(T)+{}^*\chi_m.$$

As the base characters of $k + T\,{}^*H$ do not depend on the choice of the elements $X_1, X_2, \ldots, X_m$, the numbers ${}^*\chi_i$ depend only on T and ${}^*H$. We are going to denote them by ${}^*\chi_i = {}^*\chi(T, {}^*H)$.

In a similar manner the characters of $k + T\,{}^*H$ are obtained from those of ${}^*H$ *(p 277)* by the expressions

$$w(T),\ \chi_1 + w(T),\ \chi_2 + w(T), \ldots,\ \chi_\ell + w(T),\ \text{for } w(T) \neq \chi_1, \chi_2, \ldots, \chi_\ell,$$

and

$$w(T), \chi_1+w(T), \dots, \chi_{j-1}+w(T), \chi_{j+1}+w(T), \dots, \chi_\ell+w(T), \text{ for } w(T)=\chi_j,$$

where we denote the characters of $^*H$ by $\chi_1, \chi_2, \cdots \chi_\ell$.

In particular in the case when all the characters of $^*H$ are also its base characters, all the characters of $k+T\,^*H$ are also its base characters if $w(T)$ is a character of $^*H$ or if $\chi(T, {}^*H)$ is infinite.

Remark. ρ being an arbitrary element of $W(^*H)$, we can always choose an element T of order $w(T) = \rho$ of $^*H$, in such a way that $\chi(T, {}^*H)$ is equal to one of the numbers $^*\chi_1, {}^*\chi_2, \dots, {}^*\chi_m, {}^*\chi_{m+1} = \infty$ which exceeds ρ, provided that ρ is different from the numbers $^*\chi_i$. Suppose in fact that ρ is distinct from the numbers $^*\chi_1 < {}^*\chi_2 < \cdots < {}^*\chi_m$ and let $^*\chi_\ell$ be such that we have $^*\chi_\ell < \rho < {}^*\chi_{\ell+1}$. If $X_1, X_2, \dots, X_m$ is a base of $^*H$, the canonical closure of $k[X_1, X_2, \dots, X_\ell]$ contains, by definition elements of orders ρ. Let T' be one of these elements, and set $T = T' + X_h$ (with $h > \ell,\ X_{m+1} = 0$). For $\ell \le j < h-1$ the sets

$$\overline{k[X_1, X_2, \dots, X_j, T]} \mod t^{^*\chi_h}, \quad \overline{k[X_1, X_2, \dots, X_j, T']} \mod t^{^*\chi_h},$$
$$\overline{k[X_1, X_2, \dots, X_j]} \mod t^{^*\chi_h}$$

being identical, the ring $\overline{k[X_1, X_2, \dots, X_j, T]}$ does not have elements of order $w(X_{j+1}) = {}^*\chi_{j+1}$. For $j < \ell$, $\rho = w(T)$ being greater than $^*\chi_{j+1}$, the sets

$$\overline{k[X_1, X_2, \dots, X_j]} \mod t^{^*\chi_{j+1}+1}, \quad \overline{k[X_1, X_2, \dots, X_j, T]} \mod t^{^*\chi_{j+1}+1}$$

are identical and consequently $\overline{k[X_1, X_2, \dots, X_j, T]}$ does not contain elements of order $w(X_{j+1})$. However the ring

$$\overline{k[X_1, X_2, \dots, X_{h-1}, T]},$$

which contains the element T', contains also the element $T - T' = X_h$. We then have $\chi(T, {}^*H) = {}^*\chi_h$.

Let us now consider a canonical semigroup

$$^*G = {}^*G_0 = \{0, \nu_1, \nu_1+\nu_2, \dots, \nu_1+\nu_2+\cdots+\nu_{N-1}+\mathbb{N}\nu\} \quad (\nu_{N-1} \neq \nu).$$

The semigroup

$$^*G_{N-1} = \mathbb{N}\nu$$

clearly has only one character which is $\chi_1^{(N-1)} = \nu$. The characters of

$$^*G_{N-2} = \{0, \nu_{N-1} + \mathbb{N}\nu\}$$

are then, after the rule indicated above,

$$\chi_1^{(N-2)} = \nu_{N-1}, \quad \chi_2^{(N-2)} = \nu_{N-1} + \nu.$$

The characters of $^*G_{N-3}$ are obtained from the previous ones according to the same rule: *(p 278)*

$$\left.\begin{aligned} \chi_1^{(N-3)} = \nu_{N-2}, \quad \chi_2^{(N-3)} &= \nu_{N-2} + \nu_{N-1}, \\ \chi_3^{(N-3)} &= \nu_{N-2} + \nu_{N-1} + \nu, \end{aligned}\right\} \quad \text{for} \quad \nu_{N-2} > \nu_{N-1} + \nu,$$

$$\chi_1^{(N-3)} = \nu_{N-2}, \quad \chi_2^{(N-3)} = \nu_{N-2} + \nu_{N-1}, \qquad \text{for} \quad \nu_{N-2} = \nu_{N-1} + \nu$$

$$\chi_1^{(N-3)} = \nu_{N-2}, \quad \chi_2^{(N-3)} = \nu_{N-2} + \nu_{N-1} + \nu, \qquad \text{for} \quad \nu_{N-2} = \nu_{N-1}.$$

We obtain successively, by applying always the same rule, the characters

$$\chi_1^{(i)}, \chi_2^{(i)}, \dots, \chi_{\ell_i}^{(i)}$$

of all the semigroups $\quad ^*G_i = \{0, \nu_{i+1} + G_{i+1}\}$.

Now let

$$\begin{aligned} &^*\ell_{N-1} = 1, \qquad {}^*\chi_1^{(N-1)} = \nu; \\ &^*\ell_{N-2} = 2, \qquad {}^*\chi_1^{(N-2)} = \nu_{N-1}, \qquad {}^*\chi_2^{(N-2)} = \nu_{N-1} + \nu; \end{aligned}$$

and in general

$$\begin{aligned} {}^*\ell_{i-1} = {}^*\ell_i, \quad {}^*\chi_1^{(i-1)} = \nu_i,\ {}^*\chi_2^{(i-1)} = \nu_i + {}^*\chi_1^{(i)}, \dots, {}^*\chi_{h_i}^{(i-1)} = \nu_i + {}^*\chi_{{}^*h_i-1}^{(i)}, \\ {}^*\chi_{{}^*h_i+1}^{(i-1)} = \nu_i + {}^*\chi_{{}^*h_i+1}^{(i)}, \dots, {}^*\chi_{{}^*\ell_{i-1}}^{(i-1)} = \nu_i + {}^*\chi_{{}^*\ell_i}^{(i)}, \text{ for } {}^*h_i \leq {}^*\ell_i, \\ {}^*\ell_{i-1} = {}^*\ell_i + 1, \quad {}^*\chi_1^{(i-1)} = \nu_i,\ {}^*\chi_2^{(i-1)} = \nu_i + {}^*\chi_1^{(i)}, \dots, {}^*\chi_h^{(i-1)} = \nu_i + {}^*\chi_{h-1}^{(i)}, \\ {}^*\chi_{h+1}^{(i-1)} = \nu_i + {}^*\chi_h^{(i)}, \dots, {}^*\chi_{{}^*\ell_{i-1}}^{(i-1)} = \nu_i + {}^*\chi_{{}^*\ell_i}^{(i)}, \text{ for } {}^*h_i = {}^*\ell_i + 1, \end{aligned}$$

where $^*h_i$ is any of the positive integers $h \leq {}^*\ell_i + 1$ for which we have $\nu_i < {}^*\chi_h^{(i)}$ with ${}^*\chi_{{}^*\ell_i+1}^{(i)} = \infty$, if $\nu_i \neq {}^*\chi_1^{(i)}, \dots, {}^*\chi_{{}^*\ell_i}^{(i)}$; if not ${}^*\chi_{{}^*h_i}^{(i)}$ is the one among ${}^*\chi_1^{(i)}, {}^*\chi_2^{(i)}, \dots, {}^*\chi_{{}^*\ell_i}^{(i)}$ which is equal to ν_i.

It follows immediately from the preceding remarks and the considerations before them that we can always choose the elements $T_i \in {}^*H_i$ in such a manner that the characters and the base characters of the rings

$$\begin{array}{ll}
{}^*H_{N-1} = k[T], & w(T) = \nu, \\
{}^*H_{N-2} = k + {}^*H_{N-1}T_{N-1}, & w(T_{N-1}) = \nu_{N-1} \\
\cdots\cdots & \cdots\cdots \\
{}^*H_{i-1} = k + {}^*H_iT_i, & w(T_i) = \nu_i, \\
\cdots\cdots & \cdots\cdots \\
{}^*H = {}^*H_0 = k + {}^*H_1T_1, & w(T_1) = \nu_1
\end{array}$$

are respectively

$$\begin{array}{ll}
\text{The characters} & \text{The base characters} \\
\chi_1^{(N-1)}; & {}^*\chi_1^{(N-1)}; \\
\chi_1^{(N-2)}, \chi_2^{(N-2)}; & {}^*\chi_1^{(N-2)}\, {}^*\chi_2^{(N-2)}; \\
\cdots\cdots & \cdots\cdots \\
\chi_1^{(i-1)}, \chi_2^{(i-1)}, \ldots, \chi_{\ell_{i-1}}^{(i-1)}; & {}^*\chi_1^{(i-1)}, {}^*\chi_2^{(i-1)}, \ldots, {}^*\chi_{{}^*\ell_{i-1}}^{(i-1)}; \\
\cdots\cdots & \cdots\cdots \\
\chi_1^{(0)}, \chi_2^{(0)}, \ldots, \chi_{\ell_0}^{(0)}; & {}^*\chi_1^{(0)}, {}^*\chi_2^{(0)}, \ldots, {}^*\chi_{{}^*\ell_0}^{(0)}.
\end{array}$$

In particular the base characters of ${}^*H = {}^*H_0$ coincide with its characters if (p 279)
and only if we choose ${}^*h_i = {}^*\ell_i + 1$ every time we had to make a choice; the dimension of ${}^*H$ will then be the greatest of the dimensions of the canonical rings having the same characters.

Theorem 5. *If the base characters*

$${}^*\chi_1^{(N-1)}; {}^*\chi_1^{(N-2)}, {}^*\chi_2^{(N-2)}; \ldots; {}^*\chi_1^{(i-1)}, {}^*\chi_2^{(i-1)}, \ldots, {}^*\chi_{{}^*\ell_{i-1}}^{(i-1)}, \ldots; {}^*\chi_1^{(0)}, \ldots, {}^*\chi_{{}^*\ell_0}^{(0)}$$

are constructed by setting

$${}^*\chi_{{}^*h_j}^{(j)} = \textit{ the smallest of the numbers } {}^*\chi_1^{(j)}, {}^*\chi_2^{(j)}, \ldots, {}^*\chi_{{}^*\ell_j+1}^{(j)}$$
$$\textit{which are not less than } \nu_j,$$

the dimension of the ring corresponding to ${}^*H$ *is the smallest possible among the dimensions of canonical rings having the same characters.*

Proof. Let

$$^{\dagger}\chi_1^{(N-1)};\, ^{\dagger}\chi_1^{(N-2)},\, ^{\dagger}\chi_2^{(N-2)};\ldots;\, ^{\dagger}\chi_1^{(i-1)},\, ^{\dagger}\chi_2^{(i-1)},\ldots,\, ^{\dagger}\chi_{^{\dagger}\ell_{i-1}}^{(i-1)};\ldots$$

be another system of base characters, obtained from the same numbers ν_j. We have to show that we have $^{\dagger}\ell_i \geq {}^{*}\ell_i$ $(i = N-1, N-2, \ldots, 0)$. ν being an arbitrary integer, denote by $^{*}\ell_i(\nu)$ the number of those

$$^{*}\chi_1^{(i)},\, ^{*}\chi_2^{(i)},\ldots,\, ^{*}\chi_{^{*}\ell_i}^{(i)}$$

which are not smaller than ν. Similarly let $^{\dagger}\ell_i(\nu)$ be the number of those $^{\dagger}\chi_1^{(i)}$, $^{\dagger}\chi_2^{(i)},\ldots,\, ^{\dagger}\chi_{^{\dagger}\ell_i}^{(i)}$ which are not smaller than ν. We will prove, at the same time, that we have

$$^{\dagger}\ell_i(\nu) - {}^{*}\ell_i(\nu) \leq {}^{\dagger}\ell_i - {}^{*}\ell_i.$$

The equalities

$$^{\dagger}\ell_{N-1} = {}^{*}\ell_{N-1} = 1, \quad ^{\dagger}\ell_{N-2} = {}^{*}\ell_{N-2} = 2,$$
$$^{\dagger}\ell_{N-1} - {}^{*}\ell_{N-1} = {}^{\dagger}\ell_{N-1}(\nu) - {}^{*}\ell_{N-1}(\nu) = 0,$$
$$^{\dagger}\ell_{N-2} - {}^{*}\ell_{N-2} = {}^{\dagger}\ell_{N-2}(\nu) - {}^{*}\ell_{N-2}(\nu) = 0$$

being obvious, it suffices to conclude from

$$^{\dagger}\ell_i \geq {}^{*}\ell_i, \quad ^{\dagger}\ell_i(\nu) - {}^{*}\ell_i(\nu) \leq {}^{\dagger}\ell_i - {}^{*}\ell_i$$

the inequalities

$$^{\dagger}\ell_{i-1} \geq {}^{*}\ell_{i-1}, \quad ^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) \leq {}^{\dagger}\ell_{i-1} - {}^{*}\ell_{i-1}.$$

We distinguish the following cases:

(1) $^{\dagger}\ell_i = {}^{*}\ell_i$, $\quad ^{\dagger}\chi_{^{\dagger}h_i}^{(i)}$ is finite;

(2) $^{\dagger}\ell_i \geq {}^{*}\ell_i$, $\quad ^{\dagger}\chi_{^{\dagger}h_i}^{(i)}$ is infinite, $\quad ^{*}\chi_{^{*}h_i}^{(i)}$ is finite;

(3) $^{\dagger}\ell_i \geq {}^{*}\ell_i$, $\quad ^{\dagger}\chi_{^{\dagger}h_i}^{(i)}$ is infinite, $\quad ^{*}\chi_{^{*}h_i}^{(i)}$ is infinite;

(4) $^{\dagger}\ell_i > {}^{*}\ell_i$, $\quad ^{\dagger}\chi_{^{\dagger}h_i}^{(i)}$ is finite, $\quad ^{*}\chi_{^{*}h_i}^{(i)}$ is infinite;

(5) $^{\dagger}\ell_i > {}^{*}\ell_i$, $\quad ^{\dagger}\chi_{^{\dagger}h_i}^{(i)}$ is finite, $\quad ^{*}\chi_{^{*}h_i}^{(i)}$ is finite.;

(1) $^{\dagger}\chi_{^{\dagger}h_i}^{(i)}$ being finite, $^{\dagger}\ell_i(\nu_i)$ is not zero. $^{\dagger}\ell_i(\nu_i) - {}^{*}\ell_i(\nu_i)$ being less than or equal *(p 280)* to ${}^{\dagger}\ell_i - {}^{*}\ell_i = 0$ the number ${}^{*}\ell_i(\nu_i)$ is not zero. Then ${}^{*}\chi^{(i)}_{{}^{*}h_i}$ is finite. It follows that we have

$$ {}^{\dagger}\ell_{i-1} = {}^{\dagger}\ell_i = {}^{*}\ell_i = {}^{*}\ell_{i-1}. $$

Let us show that we still have

$$ {}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) \le {}^{\dagger}\ell_{i-1} - {}^{*}\ell_{i-1}\ (= 0) $$

for all ν. According to the recursive formulas

$$ \begin{aligned} &{}^{\dagger}\chi^{(i-1)}_1 = \nu_i,\ {}^{\dagger}\chi^{(i-1)}_2 = \nu_i + {}^{\dagger}\chi^{(i)}_1, \dots, {}^{\dagger}\chi^{(i-1)}_{{}^{\dagger}h_i} = \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i-1}, \\ &\qquad {}^{\dagger}\chi^{(i-1)}_{{}^{\dagger}h_i+1} = \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i+1}, \dots, {}^{\dagger}\chi^{(i-1)}_{{}^{\dagger}\ell_{i-1}} = \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}\ell_i}, \\ &{}^{*}\chi^{(i-1)}_1 = \nu_i,\ {}^{*}\chi^{(i-1)}_2 = \nu_i + {}^{*}\chi^{(i)}_1, \dots, {}^{*}\chi^{(i-1)}_{{}^{*}h_i} = \nu_i + {}^{*}\chi^{(i)}_{{}^{*}h_i-1}, \\ &\qquad {}^{*}\chi^{(i-1)}_{{}^{*}h_i+1} = \nu_i + {}^{*}\chi^{(i)}_{{}^{*}h_i+1}, \dots, {}^{*}\chi^{(i-1)}_{{}^{*}\ell_{i-1}} = \nu_i + {}^{*}\chi^{(i)}_{{}^{*}\ell_i}; \end{aligned} $$

it is clear that we have

$$ \begin{aligned} &{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i, && \text{for} \quad \nu \le \nu_i, \\ &{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu-\nu_i) - 1, && \text{for} \quad \nu_i < \nu \le \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}, \\ &{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu-\nu_i), && \text{for} \quad \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i} < \nu, \\ &{}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i, && \text{for} \quad \nu \le \nu_i, \\ &{}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i(\nu-\nu_i) - 1, && \text{for} \quad \nu_i < \nu \le \nu_i + {}^{*}\chi^{(i)}_{{}^{*}h_i}, \\ &{}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i(\nu-\nu_i), && \text{for} \quad \nu_i + {}^{*}\chi^{(i)}_{{}^{*}h_i} < \nu. \end{aligned} $$

It follows that, for

$$ \nu \le \nu_i + \min({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}, {}^{*}\chi^{(i)}_{{}^{*}h_i}) \quad \text{and for} \quad \nu > \nu_1 + \max({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}, {}^{*}\chi^{(i)}_{{}^{*}h_i}), $$

we have

$$ {}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu-\nu_i) - {}^{*}\ell_i(\nu-\nu_i) \le 0. $$

If ${}^{*}\chi^{(i)}_{{}^{*}h_i} < {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$, we have $\min({}^{*}\chi^{(i)}_{{}^{*}h_i}, {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}) = {}^{*}\chi^{(i)}_{{}^{*}h_i}$, $\max({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}, {}^{*}\chi^{(i)}_{{}^{*}h_i}) = {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ and

$$ {}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu-\nu_i) - {}^{*}\ell_i(\nu-\nu_i) - 1 < 0 $$
$$ (\text{for } \nu_i + {}^{*}\chi^{(i)}_{{}^{*}h_i} < \nu \le \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}). $$

If ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i} < {}^{*}\chi^{(i)}_{{}^{*}h_i}$, ν_i being less than or equal to ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$, there is no number ${}^{*}\chi^{(i)}_{j}$ between ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ and ${}^{*}\chi^{(i)}_{{}^{*}h_i}$. We then have for $\nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i} < \nu \leq \nu_i + {}^{*}\chi^{(i)}_{{}^{*}h_i}$

$$\begin{aligned} {}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) &= {}^{\dagger}\ell_i(\nu - \nu_i) - {}^{*}\ell_i(\nu - \nu_i) + 1 \\ &= {}^{\dagger}\ell_i(\nu - \nu_i) - {}^{*}\ell_i({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}) + 1 \\ &< {}^{\dagger}\ell_i({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}) - {}^{*}\ell_i({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}) + 1 \leq 1. \end{aligned}$$

(2) ${}^{\dagger}\ell_i \geq {}^{*}\ell_i$, ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ is infinite ${}^{*}\chi^{(i)}_{{}^{*}h_i}$ is finite. In this case we obviously have ${}^{\dagger}\ell_{i-1} = {}^{\dagger}\ell_i + 1$, ${}^{*}\ell_{i-1} = {}^{*}\ell_i$, and therefore ${}^{\dagger}\ell_{i-1} > {}^{*}\ell_{i-1}$. The recurrence formulas *(p 281)* which provide the numbers ${}^{\dagger}\chi^{(i-1)}_{j}$ and ${}^{*}\chi^{(i-1)}_{j}$ leads to others where we have

$$\begin{aligned} &{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i + 1, && \text{for} \quad \nu \leq \nu_i, \\ &{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu - \nu_i), && \text{for} \quad \nu_i < \nu, \\ &{}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i, && \text{for} \quad \nu \leq \nu_i, \\ &{}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i(\nu - \nu_i) - 1, && \text{for} \quad \nu_i < \nu \leq \nu_i + {}^{*}\chi^{(i)}_{{}^{*}h_i}, \\ &{}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i(\nu - \nu_i), && \text{for} \quad \nu_i + {}^{*}\chi^{(i)}_{{}^{*}h_i} < \nu, \end{aligned}$$

from which we easily obtain the inequality

$${}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) \leq {}^{\dagger}\ell_{i-1} - {}^{*}\ell_{i-1} \leq 1.$$

(3) For ${}^{\dagger}\ell_i \geq {}^{*}\ell_i$, ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ infinite, ${}^{*}\chi^{(i)}_{{}^{*}h_i}$ infinite, it is clear that we have ${}^{\dagger}\ell_{i-1} = {}^{\dagger}\ell_i + 1$, ${}^{*}\ell_{i-1} = {}^{*}\ell_i + 1$ and hence ${}^{\dagger}\ell_{i-1} \geq {}^{*}\ell_{i-1}$. The recurrence formulas which give the numbers ${}^{\dagger}\chi^{(i-1)}_{j}$, ${}^{*}\chi^{(i-1)}_{j}$ produce on the other hand

$$\begin{aligned} &{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i + 1, && {}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i + 1, && \text{for} \quad \nu \leq \nu_i, \\ &{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu - \nu_i), && {}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i(\nu - \nu_i), && \text{for} \quad \nu_i < \nu, \end{aligned}$$

from which we get

$${}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) \leq {}^{\dagger}\ell_{i-1} - {}^{*}\ell_{i-1}.$$

(4) ${}^{\dagger}\ell_i > {}^{*}\ell_i$, ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ finite, ${}^{*}\chi^{(i)}_{{}^{*}h_i}$ infinite. We then have

$${}^{\dagger}\ell_{i-1} = {}^{\dagger}\ell_i, \quad {}^{*}\ell_{i-1} = \ell_i + 1,$$

$$
\begin{aligned}
&{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i, && {}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i + 1, && \text{for} \quad \nu \le \nu_i,\\
&{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu - \nu_i) - 1, && {}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i(\nu - \nu_i), && \text{for} \quad \nu_i < \nu \le \nu_1 + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i},\\
&{}^{\dagger}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu - \nu_i), && {}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i(\nu - \nu_i), && \text{for} \quad \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i} < \nu,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&{}^{\dagger}\ell_{i-1} \ge {}^{*}\ell_{i-1},\\
&{}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) = {}^{\dagger}\ell_{i-1} - {}^{*}\ell_{i-1}, && \text{for} \quad \nu \le \nu_i,\\
&{}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) = {}^{\dagger}\ell_i(\nu - \nu_i) - {}^{*}\ell_i(\nu - \nu_i) - 1, && \text{for} \quad \nu_i < \nu \le \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i},\\
&\qquad\qquad\qquad\qquad \le {}^{\dagger}\ell_{i-1} - {}^{*}\ell_{i-1},
\end{aligned}
$$

${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ being finite but greater than or equal to ν_i while ${}^{*}\chi^{(i)}_{{}^{*}h_i}$ is finite, we have

$$
{}^{*}\ell_{i-1}(\nu) = {}^{*}\ell_i(\nu - \nu_i) = 0, \quad \text{for} \quad \nu \ge \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i},
$$

and hence

$$
\begin{aligned}
{}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) = {}^{\dagger}\ell_{i-1}(\nu) &\le {}^{\dagger}\ell_{i-1}(\nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}), \quad \text{for} \quad \nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i} < \nu,\\
&\le {}^{\dagger}\ell_{i-1}(\nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}) - {}^{*}\ell_{i-1}(\nu_i + {}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i})\\
&\le {}^{\dagger}\ell_{i-1} - {}^{*}\ell_{i-1}.
\end{aligned}
$$

(5) ${}^{\dagger}\ell_i > {}^{*}\ell_i$, ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ is finite, ${}^{\dagger}\chi^{(i)}_{{}^{*}h_i}$ is finite. In this case the inequalities *(p 282)*

$$
{}^{\dagger}\ell_{i-1} \ge {}^{*}\ell_i, \quad {}^{\dagger}\ell_{i-1}(\nu) - {}^{*}\ell_{i-1}(\nu) \le {}^{\dagger}\ell_{i-1} - {}^{*}\ell_{i-1}
$$

are obtained from ${}^{\dagger}\ell_i \ge {}^{*}\ell_{i-1}$, ${}^{\dagger}\ell_i(\nu) - {}^{*}\ell_i(\nu) < {}^{\dagger}\ell_i - {}^{*}\ell_i$ in exactly the same manner as in the case (1). □

ℓ_0 being the number of characters of

$$
{}^{*}G = \{0, \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \cdots + \nu_{N-1} + \mathbb{N}\nu\},
$$

${}^{*}\ell_0$ the number of base characters ${}^{*}\chi^{(0)}_1, {}^{*}\chi^{(0)}_2, \ldots$ obtained from ${}^{*}G$ in accordance with the statement of Theorem 5, we will see that the number of base characters of a canonical ring ${}^{\dagger}H$, such that $W({}^{\dagger}H) = {}^{*}G$, is between ${}^{*}\ell_0$ and ℓ_0. Conversely one has

Theorem 6. *n being any integer between ${}^*\ell_0$ and ℓ_0, there exists a canonical ring of dimension n whose characters are those of ${}^*G$.*

Proof. It suffices to show the existence of a canonical ring of dimension n from the existence of a canonical ring of dimension $n-1$. Suppose then there exists a system of base characters

$$
{}^\dagger\chi_1^{(N-1)}; {}^\dagger\chi_1^{(N-2)}, {}^\dagger\chi_2^{(N-2)}; \dots ; {}^\dagger\chi_1^{(0)}, {}^\dagger\chi_2^{(0)}, \dots , {}^\dagger\chi_{{}^\dagger\ell_0}^{(0)}
$$

obtained from ${}^*G$ following the rules mentioned before and that we have ${}^\dagger\ell_0 = n-1$. The number ${}^\dagger\ell_0$ being smaller than ℓ_0, there exist integers i for which ${}^\dagger\chi_{{}^\dagger h_i}^{(i)}$ is finite without being equal to ν_i; let μ be the smallest of these integers. We can assume that the system of base characters

$$
{}^\dagger\chi_1^{(N-1)}; {}^\dagger\chi_1^{(N-2)}, {}^\dagger\chi_2^{(N-2)}; \dots ; {}^\dagger\chi_1^{(0)}, \dots , {}^\dagger\chi_{{}^\dagger\ell_0}^{(0)}
$$

has been chosen among the systems which satisfy the same conditions, in such a way that μ *is largest possible*. This being the case, let

$$
\begin{array}{ll}
{}^\dagger\ell'_{N-1} = {}^\dagger\ell_{N-1} = 1, & {}^\dagger\chi'^{(N-1)}_1 = {}^\dagger\chi_1^{(N-1)}, \\
{}^\dagger\ell'_{N-2} = {}^\dagger\ell_{N-2} = 2, & {}^\dagger\chi'^{(N-2)}_1 = {}^\dagger\chi_1^{(N-2)},\ {}^\dagger\chi'^{(N-2)}_2 = {}^\dagger\chi_2^{(N-2)}, \\
\dots\dots & \dots\dots \\
{}^\dagger\ell'_\mu = {}^\dagger\ell_\mu, & {}^\dagger\chi'^{(\mu)}_1 = {}^\dagger\chi_1^{(\mu)},\ {}^\dagger\chi'^{(\mu)}_2 = {}^\dagger\chi_2^{(\mu)}, \dots , {}^\dagger\chi'^{(\mu)}_{{}^\dagger\ell_\mu'} = {}^\dagger\chi^{(\mu)}_{{}^\dagger\ell_\mu'}, \\
{}^\dagger\ell'_{\mu-1} = {}^\dagger\ell_{\mu-1} + 1, & {}^\dagger\chi'^{(\mu-1)}_1 = {}^\dagger\chi_1^{(\mu-1)},\ {}^\dagger\chi'^{(\mu-1)}_2 = {}^\dagger\chi_2^{(\mu-1)}, \\
& \qquad \dots , {}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h_\mu} = {}^\dagger\chi^{(\mu-1)}_{{}^\dagger h_\mu}, \\
& {}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h_\mu+1} = \nu_\mu + {}^\dagger\chi^{(\mu)}_{{}^\dagger h_\mu},\ {}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h_\mu+2} = {}^\dagger\chi^{(\mu-1)}_{{}^\dagger h_\mu+1}, \dots
\end{array}
$$

with ${}^\dagger\chi'^{(\mu)}_{{}^\dagger h_\mu} = \infty$. The collection ${}^\dagger\chi'^{(\mu-1)}_1, {}^\dagger\chi'^{(\mu-1)}_2, \dots , {}^\dagger\chi'^{(\mu-1)}_{{}^\dagger\ell'_{\mu-1}}$ is clearly equal to the collection ${}^\dagger\chi^{(\mu-1)}_1, {}^\dagger\chi^{(\mu-1)}_2, \dots , {}^\dagger\chi^{(\mu-1)}_{{}^\dagger\ell_{\mu-1}}$ and the number ${}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h_\mu+1} = \nu_\mu + {}^\dagger\chi^{(\mu)}_{{}^\dagger h_\mu}$. The number $\nu_{\mu-1}$ cannot be equal to ${}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h_\mu+1}$. Because otherwise we would have ${}^\dagger\chi^{(\mu-1)}_{{}^\dagger h_{\mu-1}} = \infty,\ {}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h'_{\mu-1}} = {}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h_\mu+1}$ and the corresponding system

$$
\begin{array}{l}
{}^\dagger\chi'^{(\mu-2)}_1 = \nu_{\mu-1},\ {}^\dagger\chi'^{(\mu-2)}_2 = {}^\dagger\chi'^{(\mu-1)}_1 + \nu_{\mu-1}, \dots \\
\qquad {}^\dagger\chi'^{(\mu-2)}_{{}^\dagger h_\mu+1} = {}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h_\mu} + \nu_{\mu-1},\ {}^\dagger\chi'^{(\mu-2)}_{{}^\dagger h_\mu+2} = {}^\dagger\chi'^{(\mu-1)}_{{}^\dagger h_\mu+2} + \nu_{\mu-1}, \dots
\end{array}
$$

will be composed of the same numbers as the system *(p 283)*

$$^\dagger\chi_1^{(\mu-2)} = \nu_{\mu-1},\ ^\dagger\chi_2^{(\mu-2)} = {}^\dagger\chi_1^{(\mu-1)} + \nu_{\mu-1}, \ldots,$$
$$^\dagger\chi_{^\dagger h_\mu+1}^{(\mu-2)} = {}^\dagger\chi_{^\dagger h_\mu}^{(\mu-1)} + \nu_{\mu-1},\ ^\dagger\chi_{^\dagger h_\mu+2}^{(\mu-2)} = {}^\dagger\chi_{^\dagger h_\mu+1}^{(\mu-1)} + \nu_{\mu-1}, \ldots.$$

This then allows us to construct, by letting

$$^\dagger\chi_1'^{(\mu-3)} = {}^\dagger\chi_1^{(\mu-3)}, \ldots, {}^\dagger\chi'^{(\mu-3)}_{^\dagger\ell_{\mu-3}} = {}^\dagger\chi_{^\dagger\ell_{\mu-3}}^{(\mu-3)};$$
$$^\dagger\chi_1'^{(\mu-4)} = {}^\dagger\chi_1^{(\mu-4)}, \ldots, {}^\dagger\chi'^{(\mu-4)}_{^\dagger\ell_{\mu-4}} = {}^\dagger\chi_{^\dagger\ell_{\mu-4}}^{(\mu-4)};$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$^\dagger\chi_1'^{(0)} = {}^\dagger\chi_1^{(0)}, \ldots, {}^\dagger\chi'^{(0)}_{^\dagger\ell_0} = {}^\dagger\chi_{^\dagger\ell_0}^{(0)};$$

a system of base characters $^\dagger\chi_1'^{(N-1)}; \ldots; {}^\dagger\chi_1'^{(0)}, {}^\dagger\chi_2'^{(0)}, \ldots, {}^\dagger\chi_{^\dagger\ell_0}'^{(0)}$ satisfying the same conditions as the system $^\dagger\chi_1^{(N-1)}; \ldots; {}^\dagger\chi_1^{(0)}, {}^\dagger\chi_2^{(0)}, \ldots, {}^\dagger\chi_{^\dagger\ell_0}^{(0)}$ except that $^\dagger\chi'^{(i)}_{^\dagger h_i'}$ are infinite or equal to ν_i for $i = 1, 2, \ldots, \mu-1$ and μ. Therefore if $^\dagger\chi'^{(\mu')}_{^\dagger h'_{\mu'}}$ is the first of the numbers $^\dagger\chi'^{(i)}_{^\dagger h_i'}$ which is neither infinite nor equal to $\nu_{\mu'}$, we would have $\mu' > \mu$, contrary to the choice of the system

$$^\dagger\chi_1^{(N-1)}; {}^\dagger\chi_1^{(N-2)}, {}^\dagger\chi_2^{(N-2)}; \ldots; {}^\dagger\chi_1^{(0)}, {}^\dagger\chi_2^{(0)}, \ldots, {}^\dagger\chi_{^\dagger\ell_0}^{(0)}.$$

Thus $\nu_{\mu-1}$ being different than $^\dagger\chi'^{(\mu-1)}_{^\dagger h_\mu+1}$ which is the only number among $^\dagger\chi_i'^{(\mu-1)}$ which is not equal to a number $^\dagger\chi_i^{(\mu-1)}$ we can set $^\dagger\chi'^{(\mu-1)}_{^\dagger h'_{\mu-1}} = {}^\dagger\chi^{(\mu-1)}_{^\dagger h_{\mu-1}}$ and consider the set

$$^\dagger\chi_1'^{(\mu-2)} = \nu_{\mu-1},\ ^\dagger\chi_2'^{(\mu-2)} = \nu_{\mu-1} + {}^\dagger\chi_1'^{(\mu-1)}, \ldots$$

which then is composed of the numbers

$$^\dagger\chi_1^{(\mu-2)}, {}^\dagger\chi_2^{(\mu-2)}, \ldots, {}^\dagger\chi_{^\dagger\ell_{\mu-2}}^{(\mu-2)}$$

and of $^\dagger\chi'^{(\mu-1)}_{^\dagger h_\mu+1} + \nu_{\mu-1} = {}^\dagger\chi^{(\mu)}_{^\dagger h_\mu} + \nu_\mu + \nu_{\mu-1}$. Similarly we show that $\nu_{\mu-2}$ is distinct than $^\dagger\chi'^{(\mu-1)}_{^\dagger h_\mu+1} + \nu_{\mu-1}$; which allows us to set $^\dagger\chi'^{(\mu-2)}_{^\dagger h'_{\mu-2}} = {}^\dagger\chi^{(\mu-2)}_{^\dagger h_{\mu-2}}$. Continuing in this manner we finally construct the system

$$^\dagger\chi_1'^{(0)}, {}^\dagger\chi_2'^{(0)}, \ldots, {}^\dagger\chi_{^\dagger\ell_0}'^{(0)}$$

which is composed of

$$^{\dagger}\chi_1^{(0)}, {^{\dagger}\chi_2^{(0)}}, \dots, {^{\dagger}\chi_{^{\dagger}\ell_0}^{(0)}}$$

and the number $^{\dagger}\chi_{^{\dagger}h_\mu}^{(\mu)} + \nu_\mu + \nu_{\mu-1} + \cdots + \nu_1$. We then have

$$^{\dagger}\ell_0' = {^{\dagger}\ell_0} + 1 = n - 1 + 1 = n.$$

□

The following table shows the systems of base characters which correspond to the semigroup

$$^{*}G = \{0, 702, 1404, 1620, 1836, 2052, 2106, 2160, 2214, 2268, \\ 2322, 2340, 2358, 2376, 2383, 2390, 2394, 2397 + \mathbb{N}\};$$

the first column of the table being at the same time the system of characters of $^{*}G$.

	1^{st} column				2^{nd} column			3^{rd} column			4^{th} column			5^{th} column		*(p 284)*
H_{17}	1				1			1			1			1		
H_{16}	3	4			3	4		3	4		3	4		3	4	
H_{15}	4	7			4	7		4	7		4	7		4	7	
H_{14}	7	11			7	11		7	11		7	11		7	11	
H_{13}	7	18			7	18		7	18		7	18		7	18	
H_{12}	18	25			18	25		18	25		18	25		18	25	
H_{11}	18	43			18	43		18	43		18	43		18	43	
H_{10}	18	61			18	61		18	61		18	61		18	61	
H_9	54	72	115		54	72	115	54	72	115	54	72		54	72	
H_8	54	126	169		54	126	169	54	126	169	54	126		54	126	
H_7	54	180	223		54	180	223	54	180	223	54	180		54	180	
H_6	54	234	277		54	234	277	54	234	277	54	234		54	234	
H_5	54	288	331		54	288	331	54	288	331	54	288		54	288	
H_4	216	270	504	547	216	270	547	216	270	504	216	270	504	216	270	
H_3	216	486	720	763	216	486	763	216	486	720	216	486	720	216	486	
H_2	216	702	936	979	216	702	979	216	702	936	216	702	936	216	702	
H_1	702	918	1638	1681	702	918	1681	702	918	1638	702	918	1638	702	918	
H	702	1620	2340	2383	702	1620	2383	702	1620	2340	702	1620	2340	702	1620	

As examples of rings H whose characters are $702, 1620, 2340, 2383$ we can quote the following:

$$\overline{k[t^{702}, t^{1620}, t^{2340}, t^{2383}]}$$

$$\overline{k[t^{702}(1+t^{72})^3, t^{1620}(1+t^{72})^7, t^{2383}(1+t^{72})^9]}$$

$$\overline{k[t^{702}(1+t^{115})^3, t^{1620}(1+t^{115})^7, t^{2340}(1+t^{115})^9]}$$

$$\overline{k[t^{702}(1+t^{7})^{13}, t^{1620}(1+t^{7})^{30}, t^{2340}(1+t^{7})^{44}]}$$

$$\overline{k[t^{702}(1+t^{7})^{13}(1+t^{79})^3, t^{1620}(1+t^{7})^{3}(1+t^{79})^7]}$$

whose base character sequences are given by the above five columns respectively.

Finally let us point out that the characters of ${}^*H$ and the base characters of ${}^*H, {}^*H_1, \dots, {}^*H_{N-1}$ which are, as we have seen above, are invariants of ${}^*H$, do not constitute a complete system of invariants. That is to say we can construct canonical rings ${}^*H$ and ${}^*H'$ which cannot be transformed into each other by a substitution of the form

$$t \to t(\alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n + \cdots), \quad (\alpha_0 \neq 0)$$

in such a way that the characters of ${}^*H$ and ${}^*H'$, as well as the base characters of ${}^*H, {}^*H_1, \dots, {}^*H_{N-1}$ and ${}^*H', {}^*H'_1, \dots, {}^*H'_{N-1}$ are equal respectively. For example let

$$\begin{aligned} {}^*H &= k + kt^{4\nu}(1+t) + kt^{6\nu}(1+t) + kt^{7\nu}(1+t) + k[t]t^{8\nu}, \\ {}^*H' &= k + kt^{4\nu}(1+t+t^2) + kt^{6\nu}(1+t+t^2) + kt^{7\nu}(1+t+t^2) + k[t]t^{8\nu} \end{aligned}$$

where $\nu > 2$. These rings have the same characters which are

$$4\nu, 6\nu, 7\nu, 8\nu + 1.$$

Their base characters are also the same: *(p 285)*

$$4\nu, 6\nu, 7\nu.$$

The rings ${}^*H_1, {}^*H'_1$ both being identical to

$$k + kt^{2\nu} + kt^{3\nu} + k[t]t^{4\nu},$$

base characters of ${}^*H'_1, {}^*H'_2, {}^*H'_3, {}^*H'_4$ are respectively the same as those of ${}^*H_1$, ${}^*H_2, {}^*H_3, {}^*H_4$. On the other hand there exists no substitution of the form

$$(\alpha) \qquad t \to t(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots)$$

which transforms ${}^*H$ to ${}^*H'$. In fact such a transformation which maps ${}^*H$ to ${}^*H'$ should map ${}^*H_1$ to ${}^*H'_1$, i.e. onto itself. Now assuming that 2ν is not divisible by the characteristic of k, the substitutions of the form (α), which transform the ring

$${}^*H_1 = k + kt^{2\nu} + kt^{3\nu} + k[t]t^{4\nu}$$

onto itself, are of the form

$$t \to t(\alpha_0 + \alpha_\nu t^\nu + \alpha_{2\nu} t^{2\nu} + \alpha_{2\nu+1} t^{2\nu+1} + \cdots)$$

none of which transforms the element

$$t^{4\nu} + t^{4\nu+1}$$

of $^*H$ to an element of the same order in $^*H'$ which is of the form

$$\xi_0(t^{4\nu} + t^{4\nu+1} + t^{4\nu+2}) + \xi_1(t^{6\nu} + t^{6\nu+1} + t^{6\nu+2}) + \cdots.$$

Section 8:

Let us consider now an algebraic branch passing through the origin and is defined by

$$Y_1 = Y_1(t), Y_2 = Y_2(t), \ldots, Y_n = Y_n(t),$$

where $Y_1(t), Y_2(t), \ldots, Y_n(t)$ are power series in t, whose constant terms are zero. Let us consider the ring $k[Y_1(t), Y_2(t), \ldots, Y_n(t)]$. We can assume that this ring contains all elements whose orders are greater than a sufficiently large number (Lemma 2).

Theorem 7. *$^*H$ being the canonical closure of of $k[Y_1(t), Y_2(t), \ldots, Y_n(t)]$, let $W(^*H) = \{0, \nu_1, \nu_1+\nu_2, \ldots, \nu_1+\nu_2+\cdots+\nu_{N-1}+\mathbb{N}\}$. The multiplicity sequence of the successive points of the branch $Y_1(t), Y_2(t), \ldots, Y_n(t)$ is*

$$\nu_1, \nu_2, \ldots, \nu_{N-1}, 1, 1, \ldots.$$

Proof. Let $w(Y_1(t))$ be the smallest of the numbers

$$w(Y_1(t)), w(Y_2(t)), \ldots, w(Y_n(t)).$$

The point $O = (0, 0, \ldots, 0)$ is then a multiple point of order $w(Y_1(t))$. On the other hand it is clear that $w(Y_1(t)) = \nu_1$. It suffices then to show that the multiplicity sequence of the successive points $(t = 0)$ of the branch* *(p 286)*

$$Y_1'(t) = Y_1(t), Y_2'(t) = \frac{Y_2(t)}{Y_1(t)}, \ldots, Y_n'(t) = \frac{Y_n(t)}{Y_1(t)}$$

---

*See P. Du Val, loc. cit. and J. G. Semple, ”Singularities of space algebraic curves”, *Proc. London Math. Soc.* (2), 44 (1938), 149-174.

which is obtained from $Y_1(t), Y_2(t), \ldots, Y_n(t)$ by resolving it at the point O, are

$$\nu_2, \nu_3, \ldots, \nu_{N-1}, 1, 1, \ldots.$$

We move the origin of the coordinates to the point $t = 0$ of the branch $Y_1'(t), Y_2'(t), \ldots, Y_n'(t)$, which then becomes

$$Y_1'(t) - \eta_1, Y_2'(t) - \eta_2, \ldots, Y_n'(t) - \eta_n$$

where $\eta_1, \eta_2, \ldots, \eta_n$ denote the constant terms of the series $Y_1'(t), Y_2'(t), \ldots, Y_n'(t)$. I_{ν_1} being the ideal of $k[Y_1(t), Y_2(t), \ldots, Y_n(t)]$ formed by its elements of orders greater than or equal to ν_1, it is obvious that

$$[I_{\nu_1}] = k[Y_1'(t) - \eta_1, Y_2'(t) - \eta_2, \ldots, Y_n'(t) - \eta_n].$$

Now we know that

$$^*H = k + Y_1(t)\overline{[I_{\nu_1}]}$$

and that

$$W(\overline{[I_{\nu_1}]}) = \{0, \nu_2, \nu_2 + \nu_3, \ldots, \nu_2 + \nu_3 + \cdots + \nu_{N-1} + \mathbb{N}\}.$$

Therefore the origin is a multiple point of order ν_2 for the branch

$$Y_1'(t) - \eta_1, Y_2'(t) - \eta_2, \ldots, Y_n'(t) - \eta_n;$$

In other words, the smallest of the integers

$$w(Y_1'(t) - \eta_1), w(Y_2'(t) - \eta_2), \ldots, w(Y_n'(t) - \eta_n)$$

is ν_2. We complete the proof of theorem 7 by repeating this argument several times[21]. □

It follows from theorem 3 that the numbers $\nu_1, \nu_2, \ldots, \nu_{N-1}, \ldots$ are obtained from the characters of $^*H$ in exactly the same way that these numbers, considered as the multiplicities of the branch, are obtained from the characters of Du Val associated to the branch $Y_1(t), Y_2(t), Y_3(t), \ldots, Y_n(t)$. Therefore the characters of Du Val of this branch are the same as those of $k[Y_1(t), Y_2(t), \ldots, Y_n(t)]$.

It is obvious that if two branches

$$Y_1(t), Y_2(t), \ldots, Y_n(t); \quad Z_1(t), Z_2(t), \ldots, Z_m(t)$$

passing through the origin can be transformed one into the other by a birational transformation which is regular at the origin, then the rings

$$k[Y_1(t), Y_2(t), \ldots, Y_n(t)], \quad k[Z_1(t), Z_2(t), \ldots, Z_m(t)]$$

are the same or, more precisely, can be transformed one into the other by a substitution of the form $t \to t(\alpha_0 + \alpha_1 t + \cdots)$, $(\alpha_0 \neq 0)$ and conversely. We then say that these two branches are regularly equivalent two each other. For two regularly equivalent branches , the rings *(p 287)*

$$^*H = \overline{k[Y_1(t), Y_2(t), \ldots, Y_n(t)]}, \quad ^*H' = \overline{k[Z_1(t), Z_2(t), \ldots, Z_m(t)]}$$

can obviously be transformed among themselves by a substitution of the form $t \to t(\alpha_0 + \alpha_1 t + \cdots)$, $(\alpha_0 \neq 0)$; but from the identity $^*H = {}^*H'$ we cannot deduce the equality of

$$k[Y_1(t), Y_2(t), \ldots, Y_n(t)], \quad k[Z_1(t), Z_2(t), \ldots, Z_m(t)].$$

We say that the two given branches are canonically equivalent if we have $^*H = {}^*H'$. Two regularly equivalent branches are also canonically equivalent without the converse necessarily being true.The characters of $^*H$ and the base characters of $^*H_1, {}^*H_2, \ldots, {}^*H_{N-1}$ are then invariants of the branch $Y_1(t), Y_2(t), \ldots, Y_n(t)$ for canonical equivalence and consequently for regular equivalence. Let us note however that the characters and the the base characters of $^*H, {}^*H_1, {}^*H_2, \ldots, {}^*H_{N-1}$ constitute a complete system of invariants neither for canonical equivalence nor for regular equivalence; since we saw above that these characters and base characters do not suffice to determine $^*H$.

The series $Y_1(t), Y_2(t), \ldots, Y_n(t)$ clearly constitute a system of generators for $^*H = \overline{k[Y_1(t), Y_2(t), \ldots, Y_n(t)]}$.

At the end of Section 6 we saw how one can construct the system of generators of $^*H$ starting from its base elements. In particular we saw that, m being the dimension of $^*H$, i.e. the number of its base characters, every system of generators of $^*H$ contains m elements which constitute themselves a system of generators for $^*H$. This is expressed geometrically by saying that if m is the number of base characters of $k[Y_1(t), Y_2(t), \ldots, Y_n(t)]$, then one of the projections of dimension m of the branch $Y_1(t), Y_2(t), \ldots, Y_n(t)$ is canonically equivalent to it while none of the projections of dimension less than m is equivalent to $Y_1(t), Y_2(t), \ldots, Y_n(t)$.

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Translation Notes

[1]Here I will denote the number of the page where this line begins in the original text. *(page 1)*

[2]By a power series Arf always means the formal power series throughout this article. *(page 1)*

[3]Arf uses numerals to denote sections. For ease of reference I explicitly used the word **Section**. *(page 1)*

[4]It should be understood throughout the article that we always have $0 = i_0 < i_1 < i_2 < \cdots$. *(page 1)*

[5]Arf wrote *positive* here but he certainly means *non-negative*. *(page 2)*

[6]Arf uses the term *Auxiliary Theorem* but *Lemma* seems to be a better choice in English. *(page 2)*

[7]Here Arf wrote $S_i = \cdots$, but that being clearly a typo, I changed it to $S_{i_1} = \cdots$ *(page 3)*

[8]Arf does not use end-of-proof symbol but I inserted this symbol to enhance readability. *(page 4)*

[9]Canonical rings are now known as Arf rings. *(page 7)*

[10]Here Arf uses the Fraktur font $\mathfrak{G}$. I use $\mathbb{N}$. *(page 8)*

[11]This is now known as the Arf closure. *(page 10)*

[12]Arf writes *group* here but certainly means *semigroup*. *(page 10)*

[13]Here Arf does not say *integers* but it is implied. *(page 13)*

[14]Here “nonzero” is intended but is not written in the original text. *(page 18)*

[15]Arf uses $\mathfrak{H}'$ here. I use $\mathcal{H}'$. *(page 19)*

[16]Arf uses $\mathfrak{H}_{\ell-1}$ here. I use $\mathcal{H}_{\ell-1}$. *(page 21)*

[17]Here Arf uses $w(X_m)$, but $w(X_n)$ is probably *more correct*. *(page 22)*

[18] Here it is written ${}^*H_1'$ but it is a typo. I wrote ${}^*H_1$. *(page 24)*

[19]Arf wrote here $\chi_{m+1} = \infty$, but it should certainly be ${}^*\chi_{m+1} = \infty$. *(page 25)*

[20]There was a serious typo here. Instead of “*contains*”, it should be “*does not contain*”. *(page 25)*

[21]Here Arf writes ”theorem 5”, but it is clearly a typo. I wrote ”theorem 7”. *(page 39)*

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Acknowledgement

The present form of this translation owes its perfection to Nil Şahin who read an earlier version with meticulous attention to detail, caught all the errors, including mathematical ones, and made numerous suggestions for style and presentation.