In this booklet you will find solutions to the

- Fill in the boxes to make the following a true statement. No explanation is required.

type of questions which pop up from time to time in Math 101 exams.

To solve such problems in the exam you are expected to make wild and lucky guesses. With quick trial and error you can find some of these constants.

Here you will find complete solutions with explanations.

Print this manuscript as a two sided document and staple on the left long side. Then you will see the question page on the left and the answer page on the right.

Here you will find solutions to such questions that appeared in:

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Last update on 22 December 2023

Fall 2017 Midterm-1 Question-1

[17+8 points] 1

1. A function f that is defined and continuous for $x \neq 0$ satisfies the following conditions:

① $f(2-\sqrt{2}) = \sqrt[3]{1-1/\sqrt{2}}$, $f(2/3) = 0$, $f(2) = \sqrt[3]{2}$, $f(2+\sqrt{2}) = \sqrt[3]{1+1/\sqrt{2}}$

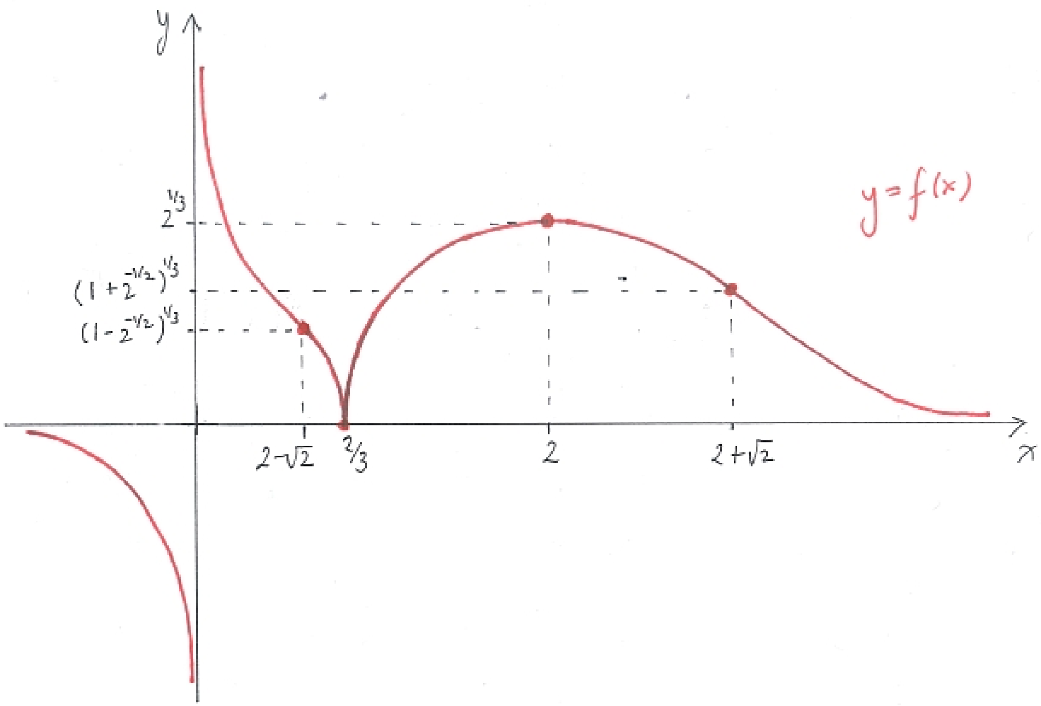
② $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 0$

③ $f'(x) < 0$ for $x < 2/3$ and $x \neq 0$, and for $x > 2$; $f'(x) > 0$ for $2/3 < x < 2$

④ $\lim_{x\to (2/3)^-} f'(x) = -\infty$, $\lim_{x\to (2/3)^+} f'(x) = \infty$

⑤ $f''(x) < 0$ for $x < 0$, and for $2-\sqrt{2} < x < 2+\sqrt{2}$ and $x \neq 2/3$; $f''(x) > 0$ for $0 < x < 2-\sqrt{2}$ and for $x > 2+\sqrt{2}$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = (ax+b)^c x^d$ satisfies the conditions ①-⑤ if a, b, c and d are chosen as

$a = \boxed{3}$, $b = \boxed{-2}$, $c = \boxed{\frac{2}{3}}$ and $d = \boxed{-1}$.

Solution of Fall 2017 Midterm-1 Question-1-b:

First observe that we cannot have $ab = 0$ since then $f'(x)$ would not change sign at $x = 2/3$. So $a \neq 0$, and $b \neq 0$.

From $f(2/3) = 0$ we get $(2a/3 + b)^c(2/3)^d = 0$. This then implies that $2a + 3b = 0$, and $c > 0$,

We have $f'(x) = ac(ax + b)^{c-1}x^d + d(ax + b)^c x^{d-1}$. From $\lim_{x\to(2/3)^+} f'(x) = +\infty$ we conclude that $a > 0$ and $0 < c < 1$.

Since $f'(x)$ is continuous at $x = 2/3$ and changes sign there we must have $f'(2/3) = 0$. Writing $f'(x) = (ax+b)^{c-1}x^{d-1}[(ac+ad)x+bd]$, and recalling that $2a+b \neq 0$, from $f'(2) = 0$ we get, after substituting $b = -2a/3$, that $3c + 2d = 0$.

Substituting $b = -2a/3$ and $c = -2d/3$, we get $f''(2 + \sqrt{2}) = (279)a^2d(d + 1)$. But $f''(x)$ is continuous at $2+\sqrt{2}$ and changes sign there, so $f''(2+\sqrt{2}) = 0$. This forces $\boxed{d = -1}$ and $\boxed{c = 2/3}$.

Now we have $f(x) = \dfrac{(a/3)^{2/3}(3x - 2)^{2/3}}{x}$. From here we have $f(2) = (a/3)^{2/3}2^{1/3}$. But $f(2) = 2^{1/3}$. Equating these and keeping in mind that $a > 0$, we get $\boxed{a = 3}$, which then gives $\boxed{b = -2}$.

Fall 2018 Midterm-2 Question-1

[17+8 points] 1

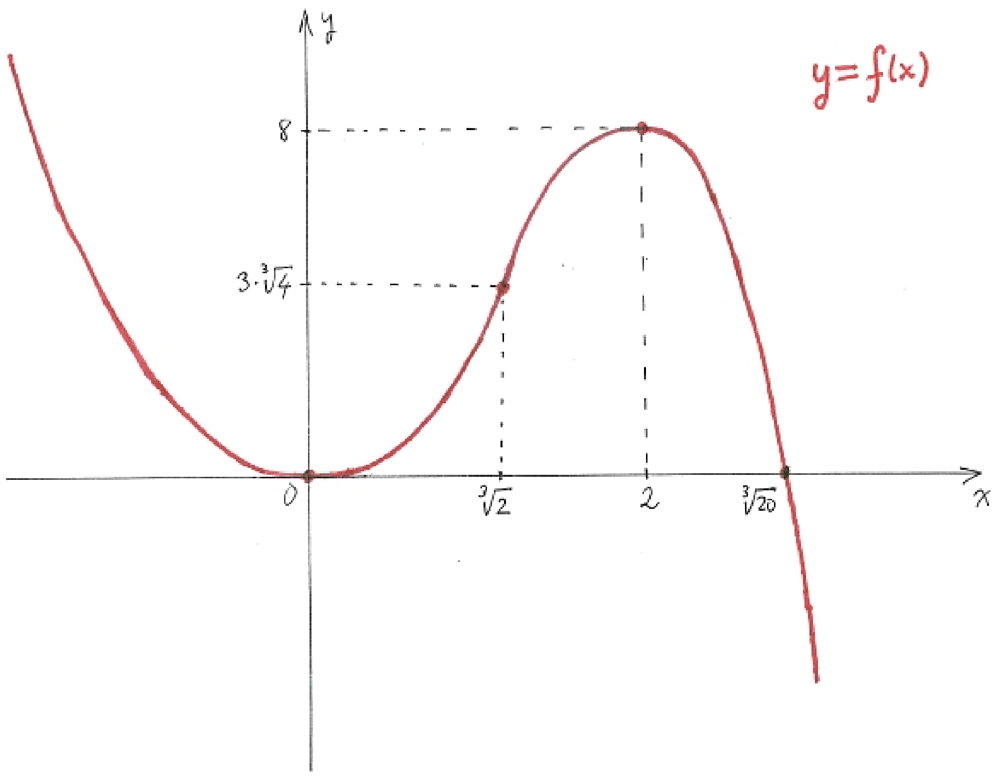
1. A function f that is twice-differentiable on the entire real line satisfies the following conditions:

❶ $f(0)=0\,,\ f(\sqrt[3]{2})=3\sqrt[3]{4}\,,\ f(2)=8\,,\ f(\sqrt[3]{20})=0$

❷ $f'(x)<0$ for $x<0$ and for $x>2$; $f'(x)>0$ for $0<x<2$

❸ $f''(x)>0$ for $x<\sqrt[3]{2}$; $f''(x)<0$ for $x>\sqrt[3]{2}$

a. Sketch the graph of $y=f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x)=ax^b+cx^d$ satisfies the conditions ❶-❸ if a, b, c and d are chosen as

$a=\boxed{-\frac{1}{6}}$, $b=\boxed{5}$, $c=\boxed{\frac{10}{3}}$ and $d=\boxed{2}$.

Solution of: Fall 2018 Midterm-2 Question-1-b:

We have $f(x) = ax^b + cx^d$, $f'(x) = abx^{b-1} + cdx^{d-1}$, and $f''(x) = ab(b-1)x^{b-2} + cd(d-1)x^{d-2}$.

If both of b and d are larger than 2, then $f''(0)$ would be zero. Since $f''(0) > 0$, we must have either $b = 2$ or $d = 2$. Let us take $\boxed{d = 2}$. (The case for $b = 2$ is similar.)

Now we have $f(x) = ax^2(x^{b-2} + \frac{c}{a})$. Since $f(20^{1/3}) = 0$, we have

$$(20^{1/3})^{b-2} = -\frac{c}{a}. \tag{1}$$

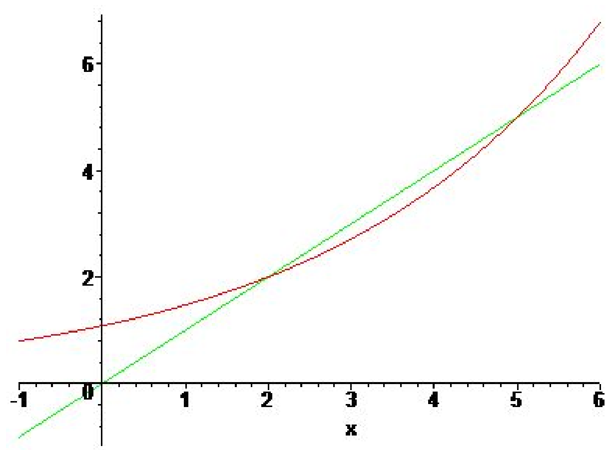
We also have $f'(x) = abx(x^{b-2} + \frac{2c}{ab})$. Since $f'(2) = 0$, we have

$$2^{b-2} = -\frac{2c}{ab}. \tag{2}$$

Dividing equation (1) by equation (2), and simplifying, we get

$$2\left(\frac{5}{2}\right)^{\frac{b-2}{3}} = b. \tag{3}$$

By inspection we see that $b = 2$ and $b = 5$ are two solutions. Since $y = 2(5/2)^{(x-2)/3}$ is concave up, and $y = x$ is a straight line, these two graphs can intersect at most at two points and we have just found all solutions. Here is the graph of these two functions:



Since b is assumed to be larger than 2, we must have $\boxed{b = 5}$.

Now equation (1) becomes

$$20 = -\frac{c}{a}, \qquad \text{or equivalently,} \qquad 20a + c = 0. \tag{3}$$

On the other hand $f(2) = 8$ gives

$$8a + c = 2. \tag{4}$$

Solving equations (3) and (4) together we get

$$\boxed{a = -\frac{1}{6}}, \text{ and } \quad \boxed{c = \frac{10}{3}}.$$

If at the beginning, instead of $d = 2$, we had chosen $b = 2$, then we would find $d = 5$, $a = 10/3$ and $c = -1/6$.

Fall 2019 Midterm-1 Question-1

[5+5+5+5+5 points] 1

1. Suppose that $\lim_{x\to 0^+} f(x) = A$, $\lim_{x\to 0^-} f(x) = B$, $f(0) = C$, where A, B, C are distinct real numbers.

In each of the following, fill in the corresponding box by:

- Expressing the limit in terms of A, B, C if it is possible to do so using the given information;
- Writing `DNE` if it is possible to conclude that the limit does not exist using the given information; or
- Putting a ✗, otherwise.

No explanation is required. No partial points will be given. [The box should contain nothing except your answer!]

a. $\lim_{x\to 0^+} f(x-\sqrt{x}) = \boxed{B}$

b. $\lim_{x\to 0^+} f(x\sin^2(1/x)) = \boxed{\text{DNE}}$

c. $\lim_{x\to 0^+} f(x - x^2\sin(1/x)) = \boxed{A}$

d. $\lim_{x\to 0^+} f(x-\sin(x)) = \boxed{A}$

e. $\lim_{x\to 0^+} f(x-\tan(x)) = \boxed{B}$

Solution of Fall 2019 Midterm-1 Question-1-b:

Q-1-a) When $0 < x < 1$, we have $x - \sqrt{x} < 0$. If we let $t = x - \sqrt{x}$, then t goes to zero from the left while x is going to zero from the right. Hence

$$\lim_{x\to 0^+} f(x - \sqrt{x}) = \lim_{t\to 0^-} f(t) = \boxed{\mathbf{B}}.$$

Q-1-b) When $0 < x < 1$, we have $x\sin^2(1/x) \geq 0$. In fact when $x_n = 1/(\pi + 2n\pi)$ for $n = 1, 2, 3, \ldots,$ we have $f(x_n \sin^2(1/x_n) = f(0) = C$. This means that in any neighborhood $(0, \delta)$ with $\delta > 0$, we have infinitely many points where $f(x\sin^2(1/x)) = C$. When $x \to 0^+$ but $x \neq x_n$, then $x\sin^2(1/x) \to 0^+$. Hence

$$\lim_{\substack{x\to 0^+ \\ x\neq x_n}} f(x\sin^2(1/x)) = A, \text{ and } \lim_{\substack{x\to 0^+ \\ x=x_n}} f(x\sin^2(1/x)) = \lim_{n\to\infty} f(x_n\sin^2(1/x_n)) = C.$$

Since $A \neq C$, we have $\lim\limits_{x\to 0^+} f(x\sin^2(1/x))$ does not exist. Hence

$$\lim_{x\to 0^+} f(x\sin^2(1/x)) = \boxed{\mathbf{DNE}}$$

Q-1-c) When $0 < x < 1$, we have $t = x - x^2\sin(1/x) = x(1 - x\sin(1/x)) > 0$. This shows that as x goes to zero from the right, t also goes to zero from the right. Hence

$$\lim_{x\to 0^+} f(x - x^2\sin(1/x)) = \lim_{t\to 0^+} f(t) = \boxed{\mathbf{A}}.$$

Q-1-d) Putting $t = x - \sin x$, we see that for $0 < x$ we have $t > 0$, and as x goes to zero from the right, t also goes to zero from the right. Hence

$$\lim_{x\to 0^+} f(x - \sin x) = \lim_{t\to 0^+} f(t) = \boxed{\mathbf{A}}.$$

Q-1-e) Putting $t = x - \tan x$, we see that for $0 < x < \pi/2$, we have $t < 0$. [This can be seen as follows: Let $\phi(x) = x - \tan x$ for $0 < x < \pi/2$. Then $\phi(0) = 0$ but $\phi'(x) = 1 - \sec^2 x < 0$, so $\phi(x)$ is decreasing starting from $\phi(0) = 0$ and is negative on $0 < x < \pi/2$.] Also note as before that as x goes to zero from the right, then t also goes to zero but from the left. Hence

$$\lim_{x\to 0^+} f(x - \tan x) = \lim_{t\to 0^-} f(t) = \boxed{\mathbf{B}}.$$

[19+6 points] 1

1. A continuous function f on $(-\infty, \infty)$ satisfies the following conditions:

❶ $f(0) = 6\,,\ f(3) = 0$

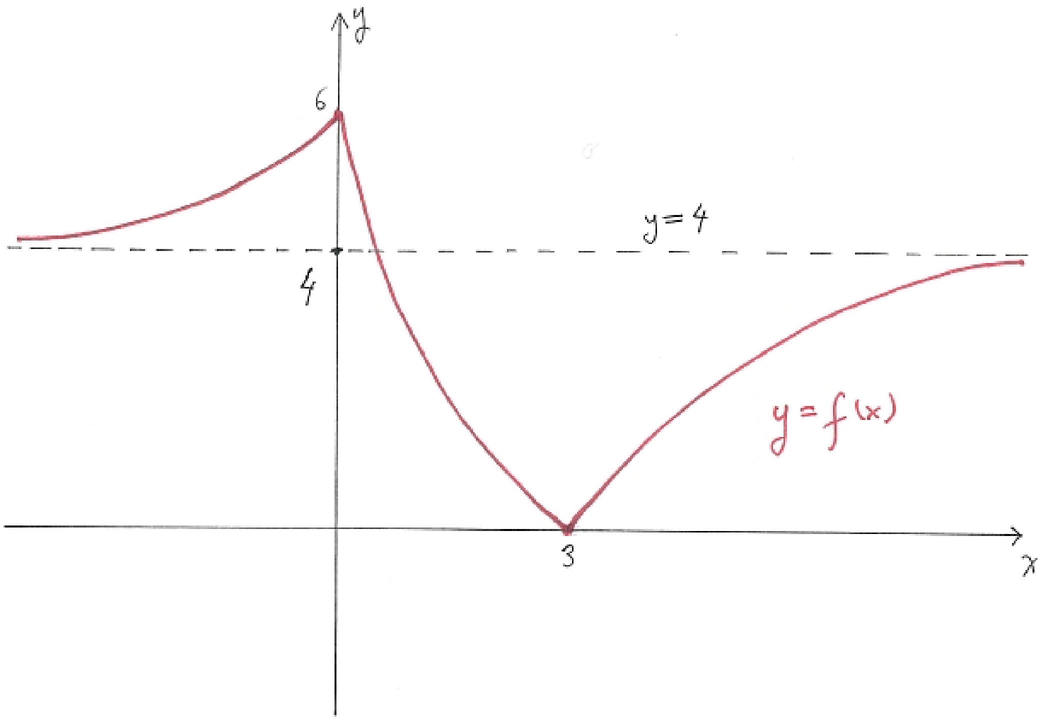
❷ $f'(x) > 0$ for $x < 0$ and for $3 < x$; $f'(x) < 0$ for $0 < x < 3$

❸ $f''(x) > 0$ for $x < 0$ and for $0 < x < 3$; $f''(x) < 0$ for $3 < x$

❹ $\lim\limits_{x\to-\infty} f(x) = 4\,,\ \lim\limits_{x\to\infty} f(x) = 4$

❺ $\lim\limits_{x\to0^-} f'(x) = 1\,,\ \lim\limits_{x\to0^+} f'(x) = -5\,;\ \lim\limits_{x\to3^-} f'(x) = -4/5\,,\ \lim\limits_{x\to3^+} f'(x) = 4/5$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = \dfrac{|x-a|}{b|x|+c}$ satisfies the conditions ❶-❺ if a, b and c are chosen as

$a = \boxed{3}$, $b = \boxed{\frac{1}{4}}$ and $c = \boxed{\frac{1}{2}}$.

Solution of Fall 2019 Midterm-2 Question-1-b

$f(3) = 0$ gives $\boxed{a = 3}$.

Now $f(0) = \dfrac{3}{c}$ but $f(0)$ is given as 6. So from $\dfrac{3}{c} = 6$ we get $\boxed{c = \dfrac{1}{2}}$.

At this point we have, for $x \geq 3$, $f(x) = \dfrac{x-3}{bx + (1/2)}$.

$\lim_{x\to\infty} f(x) = \dfrac{1}{b}$. But this limit is given as 4. So we must have $\boxed{b = \dfrac{1}{4}}$.

Fall 2020 Midterm-1 Question-5

5. A function f, which is defined and continuous for all $x \neq 2$, satisfies the following conditions:

① $f(0) = 2$, $f(\sqrt{6}-2) = (22+8\sqrt{6})/25$

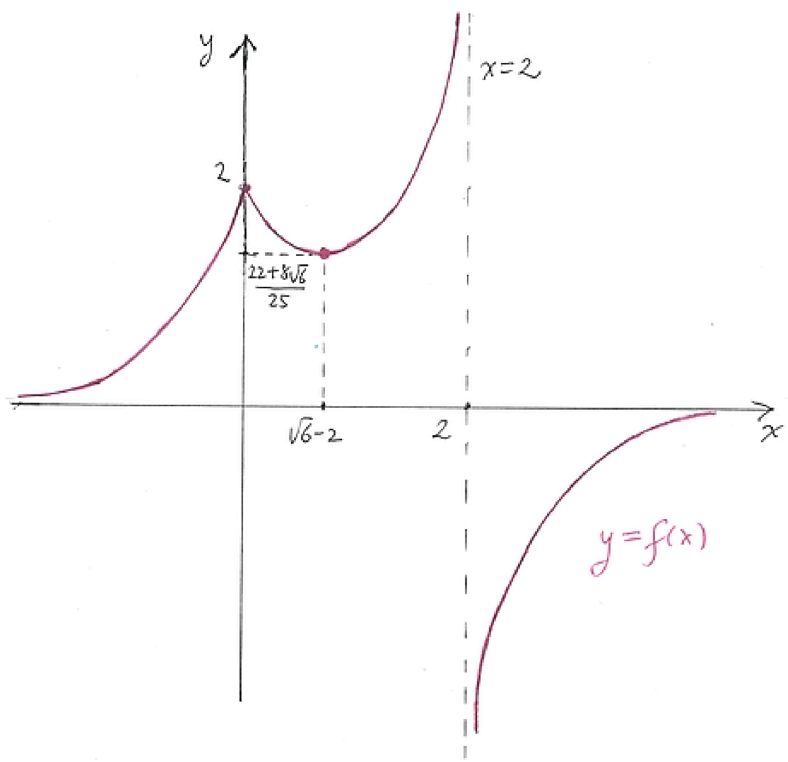
② $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 0$

③ $f'(x) > 0$ for $x < 0$, and for $x > \sqrt{6}-2$ and $x \neq 2$; and $f'(x) < 0$ for $0 < x < \sqrt{6}-2$

④ $\lim_{x\to 0^-} f'(x) = 4$, $\lim_{x\to 0^+} f'(x) = -2$

⑤ $f''(x) > 0$ for $x < 2$ and $x \neq 0$, $f''(x) < 0$ for $x > 2$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = \dfrac{ax+b}{x^2+c|x|+d}$ satisfies the conditions ①-⑤ at all points in its domain if a, b, c and d are chosen as

$a = \boxed{-1}$, $b = \boxed{-2}$, $c = \boxed{-\frac{3}{2}}$ and $d = \boxed{-1}$.

Solution of Fall 2020 Midterm-1 Question-5-b

$f(0) = 2$ gives $b = 2d$.

When $x = 2$, the denominator $x^2 + c|x| + d = x^2 + cx + d$ vanishes which gives

$$d = -4 - 2c, \quad \text{so we also get} \quad b = -8 - 4c.$$

Now we have $f(x) = \dfrac{ax - 4c - 8}{x^2 + c|x| - 4 - 2c}$.

For $0 < x < 2$ we have

$$f(x) = \frac{ax - 4c - 8}{x^2 + cx - 4 - 2c}, \quad \text{and} \quad f'(x) = -\frac{ax^2 + 4\,a + 2\,ac - 8\,cx - 4\,c^2 - 16\,x - 8\,c}{\left(x^2 + cx - 4 - 2\,c\right)^2}$$

From here we have $\displaystyle\lim_{x\to 0^+} f'(x) = \frac{2c - a}{4 + 2c}$. But we also have $\displaystyle\lim_{x\to 0^+} = -2$. Thus we get

$$a = 8 + 6c.$$

Putting this too into f we get for $x < 0$,

$$f(x) = \frac{(8 + 6c)x - 4c - 8}{x^2 - cx - 2c - 4}, \; x < 0,$$

and

$$f'(x) = -2\,\frac{4\,x^2 - 4\,cx + 16 + 24\,c + 3\,cx^2 + 8\,c^2 - 8\,x}{\left(x^2 - cx - 4 - 2\,c\right)^2}, \; x < 0.$$

Then

$$\lim_{x\to 0^-} f'(x) = \frac{-4(c+1)(c+2)}{(c+2)^2}.$$

Since this limit exists we must have $c \neq -2$, so we can cancel $(c + 2)$ to get

$$\lim_{x\to 0^-} f'(x) = \frac{-4(c+1)}{(c+2)}.$$

But this limit is given as 4. Solving for c from

$$\frac{-4(c+1)}{(c+2)} = 4,$$

we get $\boxed{c = -\dfrac{3}{2}}$.

Then substituting this value of c into the previous findings for a, b and d we get $\boxed{a = -1}$, $\boxed{b = -2}$ and $\boxed{d = -1}$.

Fall 2021 Midterm-1 Question-5

5. A twice-differentiable function f on $(-\infty, \infty)$ satisfies the following conditions:

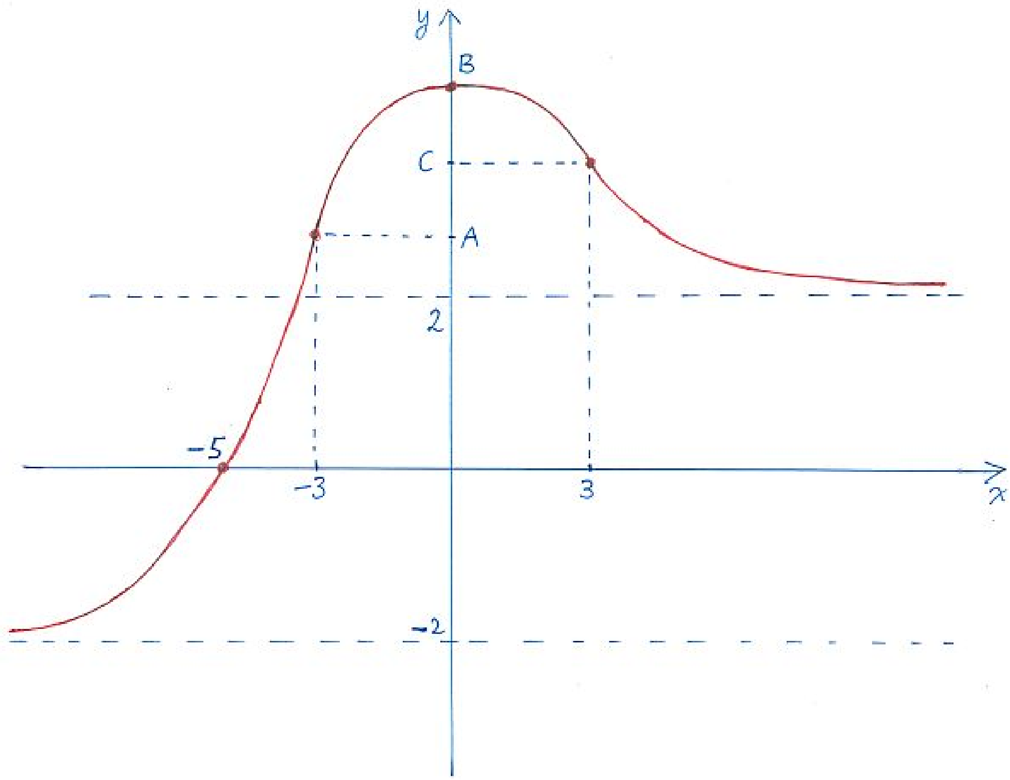
① $f(-5) = 0$, $f(-3) = A$, $f(0) = B$, $f(3) = C$, where A, B, C are real numbers such that $2 < A < C$

② $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 2$

③ $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ for $x > 0$

④ $f''(0) = 0$, $f''(x) > 0$ for $x < -3$ and for $x > 3$, $f''(x) < 0$ for $-3 < x < 0$ and for $0 < x < 3$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = \dfrac{ax^3 + b}{|x|^3 + c}$ satisfies the conditions ①-④ if a, b and c are chosen as

$a = \boxed{2}$, $b = \boxed{250}$ and $c = \boxed{54}$.

Solution of Fall 2021 Midterm-1 Question-5-b

Since $f(-5) = 0$, we must have $-125a + b = 0$.

Since $\lim_{x\to\infty} f(x) = a$ and this limit is given as 2, we have $\boxed{a = 2}$ and hence $\boxed{b = 250}$.

We now have

$$f(x) = \frac{2x^3 + 250}{|x^3| + c}.$$

Since this function is defined everywhere, the denominator never vanishes. Hence we must have $c > 0$.

Note that for $x > 0$ we have

$$f'(x) = \frac{6x^2(c - 125)}{(x^3 + c)^2} < 0.$$

Hence $c < 125$. Thus $0 < c < 125$.

We also note that for $x > 0$,

$$f''(x) = \frac{12x}{(x^3 + c)^3}[(250 - 2c)x^3 + c(c - 125)].$$

Since the denominator does not vanish for $x > 0$, this rational function is continuous and changes sign at $x = 3$. Setting $f''(3) = 0$ we end up with the equation

$$[c^2 - 179c + 6750] = [(c - 125)(c - 54)] = 0.$$

Therefore c is either 54 or 125. Since $0 < c < 125$, we conclude that $\boxed{c = 54}$.

Fall 2022 Midterm-1 Question-4

[19+6 points] 4

4. A function f, which is continuous on $[0,\infty)$ and twice-differentiable on $(0,\infty)$, satisfies the following conditions:

① $f(0)=2$, $f(3-\sqrt{5})=1+\sqrt{5}$, $f(2)=3$

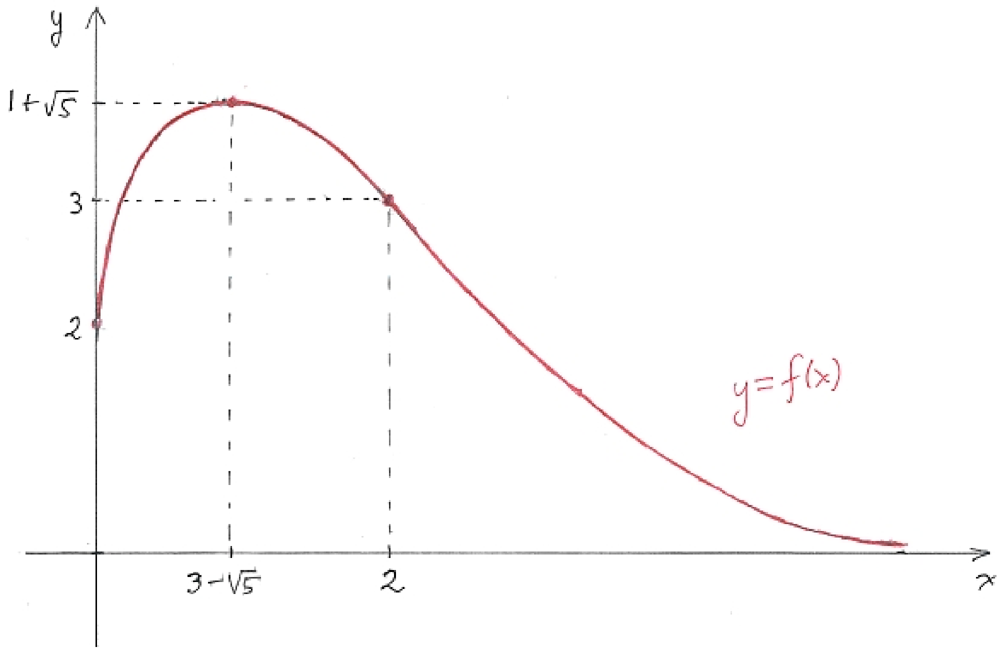
② $\lim_{x\to\infty} f(x)=0$

③ $f'(x)>0$ for $0<x<3-\sqrt{5}$, and $f'(x)<0$ for $x>3-\sqrt{5}$

④ $\lim_{x\to 0^+} f'(x)=\infty$

⑤ $f''(x)<0$ for $0<x<2$, and $f''(x)>0$ for $x>2$

a. Sketch the graph of $y=f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x)=\dfrac{a\sqrt{x}+b}{x+c}$ satisfies the conditions ①-⑤ at all points in its domain if a, b and c are chosen as

$a=\boxed{4\sqrt{2}}$, $b=\boxed{4}$ and $c=\boxed{2}$.

Solution of Fall 2022 Midterm-1 Question-4-b

$f(0) = 2$ gives $b = 2c$.

$f(2) = 3$ gives $a = \dfrac{6+c}{\sqrt{2}}$.

Then our function becomes

$$f(x) = \frac{\sqrt{2}(6+c)\sqrt{x} + 4c}{2(x+c)}.$$

Next we put $x = 3 - \sqrt{5}$ to get

$$f(3-\sqrt{5}) - (1+\sqrt{5}) = \frac{-2\sqrt{5} + 2 + c\sqrt{5} - c}{2(\sqrt{5} - 3 - c)} = 0.$$

Equating the numerator to 0 we see that $\boxed{c = 2}$.

Then from the previous equations about a and b we get $\boxed{a = 4\sqrt{2}}$, $\boxed{b = 4}$.

Fall 2023 Final Question-4

4. In each of the following, if the given statement is true for all f, then mark the ☐ to the left of TRUE with a ✗ ; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample.

a. If f has a derivative on $(-\infty, \infty)$, then f has an antiderivative on $(-\infty, \infty)$.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$ $\boxed{}$

b. If f has an antiderivative on $(-\infty, \infty)$, then f has a derivative on $(-\infty, \infty)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\boxed{|x|}$

c. If $f'(x + 2\pi) = f'(x)$ for all x, then $f(x + 2\pi) = f(x)$ for all x.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\boxed{x}$

d. If $f(n) \geq n$ for all positive integers n, then $\lim\limits_{x\to\infty} f(x) = \infty$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\boxed{x \cos(2\pi x)}$

e. If f is continuous on $(-\infty, \infty)$, then $\dfrac{d}{dx}\displaystyle\int_0^x f(xt)\, dt = f(x^2)$ for all x.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\boxed{x}$

Solution of Fall 2023 Final Question-4:

4a. If f has a derivative, then f is continuous. Every continuous function has an antiderivative by the Fundamental Theorem of Calculus. In fact $g(x) = \int_0^x f(t)\,dt$ is an antiderivative of f when f is continuous.

Hence the answer here is $\boxed{\textbf{TRUE}}$

4b. This sounds too god to be true! So we look for a counterexample. For example, we know that every continuous function has an antiderivative (see 4a. above) but not every continuous function is differentiable. Any example of a non-differentiable continuous function will serve as a counterexample.

So the answer here is $\boxed{\textbf{FALSE}}$

And a counterexample is $\boxed{|x|}$

4c. This again looks suspicious. For example if you integrate both sides of $f'(x+2\pi) = f'(x)$, then an arbitrary additive constant will come into play which will definitely change the way f behaves. For example if $f'(x) = 1$, then clearly $f'(x)$ is periodic but none of its antiderivatives, $x + C$, is periodic.

So the answer here is $\boxed{\textbf{FALSE}}$

And a counterexample is $\boxed{x}$

4d. Since no continuity condition is imposed on f we can construct an easy function as follows: $f(x) = x$ when x is an integer, and $f(x) = 0$ otherwise. Then clearly $f(x)$ has no limit as x goes to infinity.

So the answer here is $\boxed{\textbf{FALSE}}$

Another counterexample is given as $\boxed{x\cos(2\pi x)}$

4e. We met this before in question 4b of this year's second midterm exam! In fact after substituting $u = xt$ we find

$$\int_0^x f(xt)\,dt = \frac{1}{x}\int_0^{x^2} f(u)\,du.$$

Hence we get

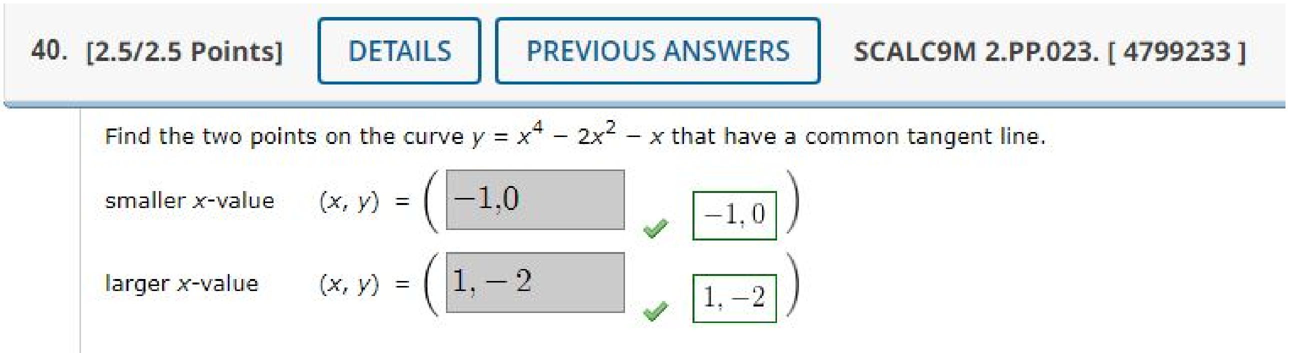
$$\frac{d}{dx}\int_0^x f(xt)\,dt = -\frac{1}{x^2}\int_0^{x^2} f(u)\,du + 2f(x^2) \stackrel{?}{=} f(x^2).$$

Clearly not every continuous function will satisfy such an elaborate identity.

So the answer here is $\boxed{\textbf{FALSE}}$

And a counterexample is $\boxed{x}$

The notorious Problem 40 of Homework-2 in Fall 2022


40. [2.5/2.5 Points] DETAILS PREVIOUS ANSWERS SCALC9M 2.PP.023. [4799233]

Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

smaller x-value $(x, y) = \left(-1,0 \quad -1, 0 \right)$

larger x-value $(x, y) = \left(1, -2 \quad 1, -2 \right)$

Solution: Let $a \neq b$ be the two points where the tangent lines coincide. Let $y = L(x)$ be an equation for the tangent line to the curve $f(x) = x^4 - 2x^2 - x$ at the point $x = a$. We need to solve simultaneously the following two non-linear equations in the two unknowns a and b.

$$f'(a) = f'(b) \tag{1}$$
$$L(b) = f(b). \tag{2}$$

The first equation says that the slopes of the tangents at the points a and b are the same. The second equation says that the tangent line at $x = a$ passes through the point $(b, f(b)$ and having slope equal to $f'(b)$ is tangent to the curve also at that point.

The first equation simplifies to

$$(a - b)(a^2 + ab + b^2 - 1) = 0.$$

Since $a \neq b$ we must have the second factor equal to zero, which gives

$$b = \frac{-a \pm \sqrt{4 - 3a^2}}{2}.$$

Using the $+$ sign for b and putting it into equation (2) above we obtain

$$3a^2(2 - 3a^2) + \sqrt{4 - 3a^2}(3a^3 - 4a) + 2 = 0.$$

Solving this we get

$$a = -1 \quad \text{and} \quad a = \frac{\sqrt{3}}{3}.$$

Substituting these into the formula for b above we get

$$b = 1 \text{ and } b = \frac{\sqrt{3}}{3}, \text{ respectively.}$$

Since we need $a \neq b$, we have only

$$a = -1 \text{ and } b = 1.$$

Using the minus sign for b we get $a = 1$ and $b = -1$. Hence $\{a, b\} = \{-1, 1\}$ is the only solution.

You can also argue as follows.

The slope of the line joining the points $(a, f(a)$ and $(b, f(b)$ must be equal to $f'(a)$, or to $f'(b)$ which is the same. Thus we can write

$$\begin{aligned} \frac{f(a)-f(b)}{a-b} &= f'(a) \\ \frac{(a^4-b^4)-2(a^2-b^2)-(a-b)}{(a-b)} &= 4a^3-4a-1 \\ (a^3+a^2b+ab^2+b^3)-2(a+b) &= 4a(a-1)(a+1). \end{aligned}$$

After multiplying both sides by $a-b \neq 0$, we get

$$(a^4-b^4)-2(a^2-b^2) = 4a(a-1)(a+1)(a-b).$$

The right hand side vanishes when $a = -1, 0, 1$. In that case

$$\begin{aligned} (a^4-b^4)-2(a^2-b^2) &= 0 \\ (a^4-b^4) &= 2(a^2-b^2) \\ (a^2-b^2)(a^2+b^2) &= 2(a^2-b^2) \\ a^2+b^2 &= 2 \\ b^2 &= 2-a^2. \end{aligned}$$

Then

$$b = \begin{cases} \pm 1 & \text{if } a = \pm 1, \\ \pm\sqrt{2}, & \text{if } a = 0. \end{cases}$$

We now check that

$$f'(0) = f'(\pm 1) = -1, \text{ and } f'(\pm\sqrt{2}) = \mp\sqrt{2} - 1 \neq -1.$$

Hence the only solution is $\{a, b\} = \{-1, 1\}$