# A generalization of the Hasse-Arf theorem*)

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## Introduction

Let $L/K$ be a totally ramified finite galois extension of a complete field $K$ with respect to a discrete valuation such that the residue class field $\bar{L}$ of $L$ is a separable extension of the residue class field $\bar{K}$ of $K$. Then a class function $\mathfrak{a}_G$ on the galois group $G$ of $L/K$ is defined which turns out to be a character of $G$ called the Artin character attached to $L/K$. The purpose of this paper is to give an explicit formula of the multiplicity $f(\chi)$ of an absolutely irreducible character $\chi$ of $G$ in $\mathfrak{a}_G$[1]. Namely we prove the following.

**Theorem 1.** *Let $L/K$ be a totally ramified finite galois extension of a complete field $K$ with respect to a discrete valuation such that the residue class field $\bar{L}$ of $L$ is separable over the residue class field $\bar{K}$ of $K$. For any absolutely irreducible character $\chi$ of $G$, let $u$ be the largest index in the sequence of the ramification groups $\{G_i\}$ of $L/K$ such that any representation of $G$ affording $\chi$ is not trivial on $G_u$, where, for the unit character of $G$, $u$ is assumed to be $-1$. Then the multiplicity $f(\chi)$ of $\chi$ in the Artin character $\mathfrak{a}_G$ attached to $L/K$ is equal to $(\varphi_{L/K}(u)+1)\chi(e)$[2], where $e$ is the unit element of $G$.*

From this we obtain the following theorem which is a generalization of the Hasse-Arf theorem[3].

**Theorem 2.** *Let $L/K$ be as above. For each jump index $u$ in the sequence of the ramification groups of $L/K$, there is an absolutely irreducible character $\chi$ of $G$ such that any representation of $G$ affording $\chi$ is not trivial on $G_u$ but trivial on $G_{u+1}$. The number $\varphi_{L/K}(u)\chi(e)$ is then an integer.*

**Preliminaries.** Let $L/K$ be as before. The normalized valuation of $L$ is designated by $\nu_L$. The function $\mathrm{i}_G$ on the galois group $G$ of $L/K$ is defined by

$$\mathrm{i}_G(s) = \nu_L(s(\alpha) - \alpha) \text{ for } s \in G,$$

where $\alpha$ is a generator of the valuation ring of $L$ over that of $K$. $\mathrm{i}_G$ is well-defined, i. e., it does not depend on the special choice of $\alpha$. For each integer $i \geqq 0$, the group $G_i = \{s \in G \mid \mathrm{i}_G(s) \geqq i+1\}$ is a normal subgroup of $G$, and the descending sequence

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1) $f(\chi)$ is the exponent of the conductor of $\chi$. For linear characters the multiplicity is well known. Cf., for instance, Serre [5], VI, § 2, Prop. 5.

2) As for the definition of $\varphi_{L/K}$, cf. Preliminaries.

3) Cf. Arf [1], Hasse [3], [4] or Serre [5], IV, § 4.

$\{G_i\}$ is the sequence of the ramification groups of $L/K$. For the sake of simplicity the definition of the function $\varphi_{L/K}(x)$ is given only for integral values $x \geqq -1$. Namely $\varphi_{L/K}(m) = \sum_{i=1}^{m} | G_i/G_0 |$ for an integer $m \geqq 1$, $\varphi_{L/K}(0) = 0$, and $\varphi_{L/K}(-1) = -1$. Herein $| H |$ stands for the order of a group $H$. If $G_u \neq G_{u+1}$, then $u$ is called a jump index in the sequence $\{G_i\}$. For each jump index $u_i$, the group $G_{u_i}$ is designated by $V_i$. Note that if $s \in V_i \setminus V_{i+1}$, then $\mathfrak{i}_G(s) = u_i + 1$. The function $\mathfrak{a}_G$ on $G$ is defined by

$$\mathfrak{a}_G(s) = \begin{cases} -\mathfrak{i}_G(s), \text{ for } s \neq e, \\ \sum_{t \neq e} \mathfrak{i}_G(t), \text{ for } s = e. \end{cases}$$

where $e$ is the unit element of $G$. The theorem of Artin[4]) states that the function $\mathfrak{a}_G$ is actually a character of $G$, the Artin character attached to $L/K$. Any representation over the complex number field affording the character $\mathfrak{a}_G$ is called the Artin representation attached to $L/K$.

## Proof of the theorem

We begin with the following elementary lemma.

**Lemma 1.** *Let $G = N_0 > N_1 > \cdots > N_r = \{e\}$ be a sequence of distinct normal subgroups of a finite group $G$, $\mathfrak{v}_i$ the character of the augmentation representation of $N_i/N_{i+1}$, and $\mathfrak{v}_i^*$ be the induced character of $\mathfrak{v}_i$ for $i = 0, 1, \ldots, r-1$. Then, being $\mathfrak{r}_G$ and $1_G$ the regular and the unit character of $G$ respectively, we have*

$$\mathfrak{r}_G = 1_G + \sum_{i=0}^{r-1} \mathfrak{v}_i^*.$$

*Proof.* To prove the assertion we compare the values taken by the characters above at each point in $G$. First, for the unit element $e$ of $G$, we have $\mathfrak{r}_G(e) = | G |$. On the other hand we have

$$\left(1_G + \sum_{i=0}^{r-1} \mathfrak{v}_i^*\right)(e) + 1 + \sum_{i=0}^{r-1} | G/N_i | (| N_i/N_{i+1} - 1) = 1 + | G/N_r | - | G/N_0 | = | G |.$$

Next let $a$ be in $N_{i_0} \setminus N_{i_0+1}$. Then $\mathfrak{v}_j^*(a) = 0$ for all $j \geqq i_0 + 1$. Hence the value taken by the right-hand side in the above formula is

$$\left(1_G + \sum_{i=0}^{i_0} \mathfrak{v}_i^*\right)(a) = 1 + \sum_{i=0}^{i_0-1} | G/N_i | (| N_i/N_{i+1} | - 1) + \sum_{x \bmod N_{i_0}} \mathfrak{v}_{i_0}(x^{-1}ax)$$

$$= 1 + \sum_{i=0}^{i_0-1} | G/N_{i+1} | - \sum_{i=0}^{i_0} | G/N_i | = 0,$$

since $\mathfrak{v}_{i_0}(b) = -1$ for any $b \in N_{i_0} \setminus N_{i_0+1}$. On the other hand $\mathfrak{r}_G(a) = 0$, hence the values taken by the characters in question coincide with each other at each point in $G$. This completes the proof.

**Corollary.** Keeping the notations and assumption in Lemma 1, let $\chi$ be any absolutely irreducible character of $G$ indifferent from $1_G$. Then there is a uniquely determined index $i$ such that $\chi$ is an irreducible constituent of $\mathfrak{v}_i^*$. The index $i$ is characterized as the largest index $i$ for which any irreducible representation of $G$ affording $\chi$ is not trivial on $N_i$. The multiplicity of $\chi$ in $\mathfrak{v}_i^*$ is equal to $\chi(e)$.

[4]) Cf. Serre [5], VI, § 2, Théorème I.

*Proof.* If we could show that, for $i < j$, $\mathfrak{v}_i^*$ and $\mathfrak{v}_j^*$ do not have any common absolutely irreducible constituent, then the rest of the assertions follows from the ordinary representation theory. Now if $\varphi$ were a common irreducible constituent of $\mathfrak{v}_i^*$ and $\mathfrak{v}_j^*$, then the restriction $\varphi_{N_j}$ would consist on one hand only of the unit character of $N_j$, on the other hand it would consists of irreducible characters of $N_j$ different from the unit character. A contradiction.

**Lemma 2.** *Let $u_0 < u_1 < \cdots < u_r$ be the set of whole jump indices in the sequence of the ramification groups of $L/K$. Further let $\mathfrak{v}_i$ be the character of the augmentation representation of $V_i/V_{i+1}$ for $i = 0, 1, \ldots, r$, where $V_{r+1}$ stands for the unit subgroup of $G$. Then, being $\mathfrak{v}_i^*$ the induced character of $\mathfrak{v}_i$, we have*

$$\mathfrak{a}_G = \sum_{i=0}^{r} (\varphi_{L/K}(u_i) + 1)\mathfrak{v}_i^* .$$

*Proof.* It suffices to show that $\mathfrak{a}_G = \mathfrak{a}_{G/V_r} + (\varphi_{L/K}(u_r) + 1)\mathfrak{v}_r^*$. Here again we compare the values taken by the above characters at each element of $G$. Let $s \neq e$ be an element of $G$. Then $\mathfrak{a}_G(s) = -\mathfrak{i}_G(s)$. To compute the value of the right-hand side at $s$, we consider the following two cases:

(i) $s \notin V_r$;

(ii) $s \in V_r$.

In the first case, let $\bar{s} \in G/V_r$ be the coset containing $s$. Then $\bar{s}$ is different from the unit element $\bar{e}$ of $G/V_r$. Hence $\mathfrak{a}_{G/V_r}(s) = -\mathfrak{i}_{G/V_r}(\bar{s})$. Now, as is known[5]),

$$\mathfrak{i}_{G/V_r}(\bar{s}) = 1/|\,V_r\,| \sum_{s' \in \bar{s}} \mathfrak{i}_G(s').$$

Each element $s'$ in $\bar{s}$ is of the form $st$ with $t \in V_r$. Being $\alpha$ a generator of the valuation ring of $L$ over that of $K$, we have $i_G(st) = \nu_L(st(\alpha) - \alpha) = \nu_L(st(\alpha) - t(\alpha) + t(\alpha) - \alpha)$. The fact that $\nu_L(st(\alpha) - t(\alpha)) = \nu_L(s(\alpha) - \alpha)$ together with the assumption that $s \notin V_r$ and $t \in V_r$, implies $\mathfrak{i}_G(s') = \mathfrak{i}_G(s)$. Hence $\mathfrak{a}_{G/V_r}(s) = -\mathfrak{i}_G(s)$. On the other hand, $s \notin V_r$ implies that $\mathfrak{v}_r^*(s) = 0$. Thus we have $\mathfrak{a}_G(s) = (\mathfrak{a}_{G/V_r} + (\varphi_{L/K}(u_r) + 1)\mathfrak{v}_r^*)(s)$.

In the case (ii), $s \in V_r$ but $s \neq e$. Hence $\mathfrak{v}_r^*(s) = -|\,G/V_r\,|$, which in turn implies

$$(\varphi_{L/K}(u_r) + 1)\mathfrak{v}_r^*(s) = -1/|\,V_r\,| \sum_{i=0}^{u_r} |\,G_i\,| = -\sum_{i=0}^{r} (u_i - u_{i-1})\,|\,V_i/V_r\,|,$$

where $u_{-1}$ is, as usual, assumed to be $-1$. Furthermore $s \in V_r$ implies that the coset $\bar{s}$ in $G/V_r$ containing $s$ is the unit element $\bar{e}$. Hence

$$\begin{aligned}\mathfrak{a}_{G/V_r}(s) = \mathfrak{a}_{G/V_r}(\bar{e}) &= \sum_{\bar{t} \neq \bar{e}} \mathfrak{i}_{G/V_r}(\bar{t}) = \sum_{i=0}^{r-1} (u_i + 1)\,(|\,V_i/V_r\,| - |\,V_{i+1}/V_r\,|) \\ &= \sum_{i=0}^{r-1} (u_i - u_{i-1})\,|\,V_i/V_r\,| - (u_{r-1} + 1),\end{aligned}$$

where $u_{-1} = -1$. Here note that $(G/V_r)_i = G_i/V_r$ for any $i \leqq u_r$[6]). Thus we obtain

$$(\mathfrak{a}_{G/V_r} + (\varphi_{L/K}(u_r) + 1)\mathfrak{v}_r^*)(s) = -(u_r - u_{r-1}) - (u_{r-1} + 1) = -(u_r + 1).$$

On the other hand the assumption that $s \in V_r$ but $s \neq e$, implies that $\mathfrak{i}_G(s) = u_r + 1$. Hence $\mathfrak{a}_G(s) = (\mathfrak{a}_{G/V_r} + (\varphi_{L/K}(u_r) + 1)\mathfrak{v}_r^*)(s)$.

[5]) Cf. Serre [5], IV, § 1, Prop. 3.

[6]) Cf. Serre [5], IV, § 1, Corollaire.

To complete the proof we have to show that the same holds for $e$. This however can be seen from the fact that $\sum_{s \in G} \mathfrak{a}_G(s) = 0$ and $\sum_{s \in G} \left(\mathfrak{a}_{G/V_r} + (\varphi_{L/K}(u_r) + 1)\mathfrak{v}_r^*\right)(s) = 0$.

Now the proof of Theorems 1 and 2 follows immediately from the above lemmata.

**Remark.** The proof of Theorem 2 given above is based on the theorem of Artin, the proof of which depends on the Hasse-Arf theorem together with some reduction techniques in the representation theory such as Brauer's theorem concerning induced characters. The statement in Theorem 2 however has, at least superficially, nothing to do with Artin's representations, and the theorem of Artin is even a direct consequence of Theorem 2. In this respect it would be of interest to try to prove the theorem without using the theorem of Artin. By the way Theorem 2 is best possible. Namely Serre gave an example of a totally ramified extension $L/K$ with the galois group isomorphic to the quaternion group such that the jump indices are 1 and 3, and $\varphi_{L/K}(3) = 3/2$ [7]).

[7]) For the detail, cf. Serre [6], Section 4.

## References

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