Continuity of multivariate rational functions

Ali Sinan Sertöz

Abstract

The limiting behavior of a multivariate rational function at its only singularity is read off from the exponents that appear in the expression of the function. We give two proofs of the result, one uses a direct approach and the other uses Lagrange multipliers method.

The behavior of a multivariable rational function at its singularities is erratic. The simplest case where we have a chance of understanding its behavior is when the denominator vanishes only at the origin. In two-variable case this rational function defines a surface which either intersects the z-axis at one point or wraps around it at the origin. To decide which-happens-when is a tricky process. For this reason not many examples float in the literature. For example how do we calculate

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^2z}{x^4+y^{12}+z^{14}}, \quad \text{or} \quad \lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^2z^2}{x^4+y^{12}+z^{14}}?$$

For a multivariable rational function whose denominator vanishes only at the origin, the continuity of this function at the origin must certainly be encoded in the exponents of the variables. The task is therefore to decipher this code, which is given by the following theorem:

Theorem: *Let $a_1, \ldots, a_N$ be non-negative integers, $m_1, \ldots, m_N$ be positive integers and $c_1, \ldots, c_N$ be positive real numbers, where $N > 1$. Then*

$$\lim_{(x_1,\ldots,x_N)\to(0,\ldots,0)} \frac{x_1^{a_1}\cdots x_N^{a_N}}{c_1x_1^{2m_1}+\cdots+c_Nx_N^{2m_N}} \text{ exists if and only if } \sum_{i=1}^{N}\frac{a_i}{2m_i} > 1.$$

Moreover, when the limit exists, then it is zero.

Remarks: Before we prove this theorem, a few remarks are in order.

• First it is easy to notice that we can take all the c_i as 1 after re-scaling; define the new coordinates as $X_i = \beta_i x_i$ where $\beta_i > 0$ and $\beta_i^{2m_i} = c_i$, $i = 1, \ldots, N$. Hence from now on we will take $c_i = 1$, $i = 1, \ldots, N$.

• It is also clear that the only influence of the a_is is to set the rate of growth of the function. Therefore they can be chosen as any non-negative real numbers provided that either we restrict the choice of the variables to non-negative values or we enter into the realm of complex numbers.

• The $N = 1$ case is totally trivial and is slightly different than the general case. In that case the limit exists if and only if $\dfrac{a_1}{2m_1} \geq 1$. When it exists, the limit is 1 when equality holds and is zero otherwise.

• For notational convenience in the proof, we define $\vec{x} = (x_1, \ldots, x_N)$, and set

$$f(\vec{x}) = \frac{\prod_{i=1}^N x_i^{a_i}}{\sum_{i=1}^N x_i^{2m_i}}.$$

We also define

$$\begin{aligned} p &= \prod_{i=1}^N m_i, \\ p_i &= p/m_i, \quad i = 1, \ldots, N. \end{aligned}$$

Proof of the theorem: **First assume that the limit exists.** In this case the limit along any path must also exist and be independent of path. For this purpose set $\lambda = (\lambda_1, \ldots, \lambda_N)$ where each $\lambda_i > 0$, $i = 1, \ldots, N$. Restricting f to the path

$$\vec{x_\lambda}(t) = (\lambda_1 t^{p_1}, \ldots, \lambda_N t^{p_N}),$$

we get

$$f(\vec{x_\lambda}(t)) = \left(\frac{\prod_{i=1}^N \lambda_i^{a_i}}{\sum_{i=1}^N \lambda_i^{2m_i}} \right) t^{(a_1 p_1 + \cdots + a_N p_N) - 2p}.$$

As $t \to 0$, this limit will exist and be independent of λ only if the power of t is strictly positive, i.e.

$$a_1 p_1 + \cdots a_N p_N - 2p > 0$$

or equivalently

$$\frac{a_1}{2m_1} + \cdots + \frac{a_N}{2m_N} > 1, \tag{*}$$

which is precisely the necessary condition we seek.

Conversely assume that the inequality (*) holds. We will show that $\lim_{\vec{x}\to 0} |f(\vec{x})| = 0$.

We will use induction on N. Clearly there is nothing to prove when $N = 1$, since then $f(x_1) = x_1^{a_1 - 2m_1}$ and (*) implies immediately that the required limit exits and is zero.

Now assume $N > 1$. Our strategy will be to restrict $|f(\vec{x})|$ to lines parallel to one of the coordinate axes, say the x_1-axis, and show that it is bounded along each such line with its maximum value going to zero as the line approaches to the origin.

First we observe that if for some j we have $\frac{a_j}{2m_j} \geq 1$, then

$$|f(\vec{x})| = |x_1^{a_1} \cdots x_j^{a_j - 2m_j} \cdots x_N^{a_N}| \frac{x_j^{2m_j}}{\sum_{i=1}^N x_i^{2m_i}} \leq |x_1^{a_1} \cdots x_j^{a_j - 2m_j} \cdots x_N^{a_N}|.$$

By the inequality (*), either $a_j - 2m_j > 0$ or $a_i > 0$ for some i other than j. Then by the sandwich theorem we have $\lim_{\vec{x}\to 0} |f(\vec{x})| = 0$.

Therefore we are reduced to the case where $0 \leq a_i < 2m_i$, $i = 1, \ldots, N$. It is clear that when (*) holds, at least one of the a_i is strictly positive. Without loss of generality assume that $0 < a_1 < 2m_1$.

At this point we quote our induction hypothesis:

$$\text{If } \frac{d_2}{2m_2} + \cdots + \frac{d_N}{2m_N} > 1, \text{then } \lim_{(x_2,\ldots,x_N)\to(0,\ldots,0)} \frac{\prod_{i=2}^N |x_i|^{d_i}}{\sum_{i=2}^N x_i^{2m_i}} = 0,$$

where $d_2, \ldots, d_N$ are non-negative integers, and $m_2, \ldots, m_N$ are positive integers.

Now for any $\vec{x} = (x_1, \ldots, x_N)$ set $\pi(\vec{x}) = (|x_2|, \ldots, |x_N|)$.

We fix $\vec{x}$ and consider the non-trivial case when $\pi(\vec{x}) \neq (0, \ldots, 0)$.

We now restrict the function $f(\vec{x})$ to the line

$$t \mapsto (t, |x_2|, \ldots, |x_N|),\ t \in [0, \infty).$$

Call the restriction of f to this line by $\phi_{\pi(\vec{x})}$;

$$\phi_{\pi(\vec{x})}(t) = f(t, |x_2|, \ldots, |x_N|) = \left(\prod_{i=2}^{N} |x_i|^{a_i} \right) \frac{t^{a_1}}{t^{2m_1} + \left(\sum_{i=2}^{N} x_i^{2m_i} \right)}, \ t \in [0, \infty).$$

Clearly $\phi_{\pi(\vec{x})}(t) \geq 0$ on its domain, $\phi_{\pi(\vec{x})}(0) = 0$ and moreover $\lim_{t \to \infty} \phi_{\pi(\vec{x})}(t) = 0$. Hence the function $\phi_{\pi(\vec{x})}(t)$ will attain its maximum value at some point, say $t_{\pi(\vec{x})} \in [0, \infty)$. We then have

$$0 \leq |f(\vec{x})| = \phi_{\pi(\vec{x})}(|x_1|) \leq \phi_{\pi(\vec{x})}(t_{\pi(\vec{x})}), \text{ for all } |x_1| \in [0, \infty).$$

It now remains to show that $\lim_{\pi(\vec{x}) \to 0} \phi_{\pi(\vec{x})}(t_{\pi(\vec{x})}) = 0$.

A direct calculation yields that $\phi_{\pi(\vec{x})}(t)$ has its maximum at

$$t_{\pi(\vec{x})} = \left(\frac{a_1}{2m_1 - a_1} \right)^{\frac{1}{2m_1}} \left(\sum_{i=2}^{N} x_i^{2m_i} \right)^{\frac{1}{2m_1}}.$$

The maximum value of $\phi_{\pi(\vec{x})}(t)$ can now be written as

$$\phi_{\pi(\vec{x})}(t_{\pi(\vec{x})}) = K \, g(\pi(\vec{x}))^{(1 - \frac{a_1}{2m_1})},$$

where K is a constant and

$$g(\pi(\vec{x})) = \frac{\prod_{i=2}^{N} |x_i|^{d_i}}{\sum_{i=2}^{N} x_i^{2m_i}},$$

where $d_i = \dfrac{a_i}{1 - \frac{a_1}{2m_1}}$, $i = 2, \ldots, N$. (Compare this with our induction hypothesis above.)

The condition (*) implies that

$$\frac{d_2}{2m_2} + \cdots + \frac{d_N}{2m_N} = \left(\frac{1}{1 - \frac{a_1}{2m_1}} \right) \left(\frac{a_2}{2m_2} + \cdots + \frac{a_N}{2m_N} \right) > 1$$

and this in turn, by the induction hypothesis, implies that

$$\lim_{\pi(\vec{x}) \to 0} \phi_{\pi(\vec{x})}(t_{\pi(\vec{x})}) = 0,$$

which completes the proof. □

We can discuss even the differentiability of such fractions:

Corollary: *Let $a_1, \ldots, a_N, m_1, \ldots, m_N$ be all positive integers and $c_1, \ldots, c_N$ be positive real numbers, where $N > 1$. Then the function*

$$f(\vec{x}) = \frac{\prod_{i=1}^{N} x_i^{a_i}}{\sum_{i=1}^{N} c_i x_i^{2m_i}}$$

is C^1 at the origin if

$$\sum_{i=1}^{N} \frac{a_i}{2m_i} > 1 + \max_{1 \le j \le N} \{\frac{1}{2m_j}\}.$$

Proof: We calculate the jth partial derivative for $j = 1, \ldots, N$ and find that

$$\left| \frac{\partial f}{\partial x_j} \right| \le \frac{|x_j|^{a_j - 1} \prod_{i=1, i \ne j}^{N} |x_i|^{a_i}}{\sum_{i=1}^{N} c_i x_i^{2m_i}} (a_j + 2m_j) \, .$$

Now apply the theorem to assure the continuity of this expression at the origin. □

A final remark: The proof of the theorem reveals that there is a distinguished path, $(\lambda_1 t^{p_1}, \ldots, \lambda_N t^{p_N})$, with the property that the limit exits if and only if it exists along this path. It is tempting to ask at this point if such a *royal path* exist for every limit problem.

Postscript (April 2014): It is communicated to me by Murad Özaydın of Oklahoma University that the method of Lagrange multipliers can also be used to give an alternate proof for this theorem. Here is my rendition of that idea.

As in the above proof we assume without loss of generality that $0 \le a_i < 2m_i$, $i = 1, \ldots, N$ and that $a_1 > 0$. Also note that if $\vec{x}$ approaches the origin along a path where $x_1 = 0$, then the limit along that path is zero. Hence if the limit of $f(\vec{x})$ as $\vec{x} \to (0, \ldots, 0)$ exists, then this limit must be zero.

Now let

$$F(\vec{x}) = x_1^{a_1} \cdots x_N^{a_N}, \text{ and } G(\vec{x}) = x_1^{2m_1} + \cdots + x_N^{2m_N}.$$

Let $R > 0$ be a real number. We pose the problem of finding the minimum and maximum of $F(\vec{x})$ subject to the condition that $G(\vec{x}) = R$. We will show that this minimum and maximum values go to zero as $R \to 0$ if and only if the condition $\sum_{i=1}^{N} \frac{a_i}{2m_i} > 1$ is satisfied. This will then prove the theorem.

Applying the Lagrange multipliers method we obtain the equalities

$$a_i x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_N^{a_N} = 2\lambda m_i x_i^{2m_i - 1}, \;\; i = 1, \ldots, N.$$

Without entering the discussion about the cases when $a_i < 1$, we multiply both sides of each equation by x_i to obtain

$$a_i F(\vec{x}) = 2\lambda m_i x_i^{2m_i}, \;\; i = 1, \ldots, N.$$

The case when any $x_i = 0$ will cause $F(\vec{x}) = 0$, and this will not give the minimum or maximum values. So we may assume for the aim of obtaining the min/max points that each x_i is different than zero. Then we can divide each equation by $2m_i x_i^{2m_i}$, and eliminate λ by writing

$$\frac{a_i}{2m_i} \frac{F(\vec{x})}{x_i^{2m_i}} = \frac{a_1}{2m_1} \frac{F(\vec{x})}{x_1^{2m_1}} \;\; i = 2, \ldots, N.$$

Since we are considering the cases when $F(\vec{x}) \neq 0$, we can cancel F from each side of these equations to obtain

$$x_i^{2m_i} = \frac{a_i m_1}{a_1 m_i} x_1^{2m_1}, \;\; i = 2, \ldots, N. \qquad (**)$$

This leads to the equation, through $G(\vec{x}) = R$,

$$x_1^{2m_1} \left(1 + \frac{m_1}{a_1} \sum_{i=2}^{N} \frac{a_i}{m_i} \right) = R.$$

Note that $\left(1 + \frac{m_1}{a_1} \sum_{i=2}^{N} \frac{a_i}{m_i} \right)$ is a nonzero constant, so set

$$\alpha = \left(1 + \frac{m_1}{a_1} \sum_{i=2}^{N} \frac{a_i}{m_i} \right)^{-1}.$$

Then we have

$$x_1^{2m_1} = \alpha R, \text{ and from } (**),\; x_i^{2m_i} = \frac{a_i m_1}{a_1 m_i} \alpha R, \;\; i = 2, \ldots, N.$$

Hence the critical points which give the minimum and maximum of $F(\vec{x})$ are of the form

$$x_1 = \pm \alpha^{\frac{1}{2m_1}} R^{\frac{1}{2m_1}} \text{ and } x_i = \pm \left(\frac{a_i m_1}{a_1 m_i} \alpha \right)^{\frac{1}{2m_i}} R^{\frac{1}{2m_i}}, \;\; i = 2, \ldots, N.$$

This shows that the absolute value of the minimum value of F at these critical points is equal to the maximal value of F, and the value of F at these critical points is given by

$$|F(\vec{x})| = |x_1^{a_1} \cdots x_N^{a_N}| = AR^{\frac{a_1}{2m_1}+\cdots+\frac{a_N}{2m_N}},$$

where

$$A = A_1 \cdots A_N, \text{ and } A_1 = |\alpha^{\frac{1}{2m_1}}|,\ A_i = |\left(\frac{a_i m_1}{a_1 m_i}\alpha\right)^{\frac{1}{2m_i}}|,\ \ i = 2, \ldots, N.$$

Now the above discussions show that the maximum value M_R of $|f(\vec{x})|$ on the surface $G(\vec{x}) = R$ is given by

$$M_R = \frac{|F(\vec{x})|}{G(\vec{x})} = AR^{\frac{a_1}{2m_1}+\cdots+\frac{a_N}{2m_N}-1}.$$

If $\sum_{i=1}^N \frac{a_i}{2m_i} = 1$, then the limit of M_R as $R \to 0$ will give a non-zero constant A. We already argued that if the limit exists then this limit has to be zero, so in this case the limit does not exist. And in fact if the limit exits then this sum, $\sum_{i=1}^N \frac{a_i}{2m_i}$ cannot be one. It is now clear that the limit exists if and only if $\sum_{i=1}^N \frac{a_i}{2m_i} > 1$, and in that case the limit is zero.

This then constitutes an interesting alternate proof for the theorem.

Bilkent University, Department of Mathematics, 06800 Ankara, Turkey.
sertoz@bilkent.edu.tr