2. ARF INVARIANT AND ITS APPLICATIONS IN TOPOLOGY

In 1941 C. Arf introduced an important invariant of quadratic forms over a field of characteristic 2 [1]. This invariant, which is known as Arf invariant in the literature, turned out be extremely important in algebraic and differential topology. It seems to be the key for the solution of several classical and fundamental problems about the topology of manifolds.

We shall try to explain some applications of Arf invariant in topology. In these applications all quadratic forms will be on $\mathbb{Z}_2$, the integers modulo 2, so we shall discuss Arf invariant only for quadratic forms over this field.

Let V be a vector space over $\mathbb{Z}_2$. A map $q : V \to \mathbb{Z}_2$ is called a quadratic form if

1) $q(0) = 0$ and,

2) the pairing (,) which is defined by

$$(x, y) = q(x + y) - q(x) - q(y)$$

is bilinear.

It can be easily seen that $(x, y) = (y, x)$ and $(x, x) = q(2x) - 2q(x) = 0$ for $x, y \in V$. In other words the bilinear form (,) is symplectic. Thus if this form is nonsingular we may find a symplectic basis i.e. a basis $a_i, b_i, i = 1, \ldots n$ for V such that $(a_i, b_j) = \delta_{ij}, (a_i, a_j) = (b_i, b_j) = 0$ [2]. Arf invariant of the quadratic form can now be defined as

$$c(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbb{Z}_2$$

This invariant is independent of the choice of the basis a_i, b_i $i = 1, 2, \ldots, n$.

A theorem due to C. Arf states that two nonsingular quadratic forms on a $\mathbb{Z}_2$ vector space V of finite dimension are equivalent if and only if they have the same Arf invariant.

Now let us see some of the important applications of Arf invariant in topology.

In 1960 M. Kervaire introduced an invariant of framed $(4k+2)$-dimensional manifolds. He showed that this invariant vanishes for 10-dimensional framed manifolds and he used this to prove the existence of a closed, triangulizable 10-dimensional manifold which does not admit any differentiable structure [5]. This invariant was nothing but the Arf invariant of a certain quadratic form over the cohomology group H^{2k+1} $(M;_2)$ of the manifold M with $_2$-coefficients (which is actually a vector space over $_2$).

In 1963 J. Milnor and M. Kervaire solved the classification problem of the "homotopy spheres" up to an equivalence relation called h-cobordism [6]. Homotopy spheres are the manifolds which have the homotopy type of a sphere. The main technique Milnor and Kervaire used was to kill the homotopy groups of certain manifolds by applying so called "spherical modifications". "The obstruction" for doing this was the Arf invariant of a certain quadratic form over $_2$ associated to the original manifold. In other words a necessary and sufficient condition for applying spherical modifications to obtain the desired manifold was the vanishing of this Arf invariant.

The classification of the manifolds having the homotopy type of a sphere is a special case of more general problem, namely classification (up to a certain equivalence relation) of smooth manifolds which are homotopy equivalent to a given manifold. An important process which is used to solve this kind of problems is "surgery".

The importance of Arf invariant in surgery theory can best be seen in the statement of "the Fundamental Surgery Theorem".

Suppose $f : M_1 \to M$ is a normal map between n-dimensional manifolds, in other words assume f has degree 1 and is covered by a bundle map $\hat{f} : v_{M_1} \to \xi$ where v_{M_1} is the normal bundle of M_1 and ξ is some bundle over M. The problem of whether f is cobordant to a homotopy equivalence is a classical problem known as the Browder-Novikov Surgery Problem. In this situation a cobordism of two normal maps from M_1 to M is a smooth $(n+1)$-dimensional manifold W together with a normal map $F : W \to M$ such that the boundary of W is the disjoint union of two different copies of M_1 and the restriction of F to different copies of M_1 are the given normal maps. Surgery is the name of the process to construct normal cobordisms.

The part of the statement of the Fundamental Surgery Theorem concerning $(4k + 2)$-dimensional manifolds says that if M is simply connected and if $k \geq 1$, then a normal map given as above is "normal cobordant" to a homotopy equivalence if and only if certain obstruction vanishes. Again this obstruction is the Arf invariant of a quadratic form over a subgroup of some cohomology group of M with $_2$ coefficients. This quadratic form is associated to M by means of the map f ([4], [7]).

Now, let us go back to the Kervaire invariant of framed manifolds (which is just the Arf invariant of a specific quadratic form associated to the manifold). In 1969 W. Browder has shown that Kervaire invariant of a framed manifold is 0 if the manifold is not $(2^{k+2} - 2)$-dimensional for some k [3]. In the same paper Browder has given some homotopy theoretical conditions for $(2^{k+2} - 2)$-dimensional framed manifolds to have Kervaire invariant 1. The problem of determining integers k for which a $(2^{k+2}-2)$-dimensional manifold have Kervaire invariant 1 is a central problem in Differential Topology and via Browder's results in Homotopy Theory. This problem, which is known as the Kervaire Invariant Problem, is still largely unsolved and it seems very likely that it will remain as a challenge to topologists in the next few decades, perhaps even longer.

References

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