## 4. ARF'S WORK IN APPLIED MATHEMATICS

Arf's main contribution to applied mathematics is in the field of elasticity theory. It is the doctoral dissertation of M. İnan ([7]) that has brought to Arf's attention the problem of designing optimal profiles for supporting structures. In his dissertation, İnan had approached the problem by experimental methods based on photoelasticity and observed that the stress profiles exhibited similarities to velocity fields of the time independent, irrotational flows of ideal, incompressible fluids. Arf undertook to provide a theoretical method for computing İnan's profiles and to explain the similarity with fluid dynamics. He pursued this programme with success in six papers written in Maryland, U.S.A. and İstanbul during the years 1947-1954. We are given to understand that Arf's fundamental ideas were set forth already in 1946, in a note presented to the 6$^e$ Congrés International de Mécanique Appliquée. The proceedings of this congress were never published, subsequently the contents of Arf's note were embodied in a paper published in 1954 ([6]) largely overlapping with the 1947 paper ([1]).

Possibly guided by the experimentally observed similarity with fluid motion, Arf has chosen complex analytic functions as his instrument. In view of the spectacular success of complex analytic functions in electrostatics and time independent fluid dynamics, their appearance in connection with problems of plane elasticity theory must be deemed a belated one. It appears that after the initial struggles occupying much of the first half of the 19th century the attention of the founders of theoretical elasticity was focused upon the relevance of the theory of biharmonic functions introduced by G.B. Airy. This was possibly the reason why a relatively long time had to elapse until the initiation of complex analytic methods by G.V. Kolosov in 1909. Many contributions were made to Kolosov's methods by N.I. Muskhelishvili who systematized the subject and brought it to textbook maturity in 1939 ([8]). In his historical survey ([9]) P.P. Teodorescu credits A.C. Stevenson with a rediscovery of the method in 1945.

Arf formulates the problem of optimal profiles as a special case of the general problem of classifying stress states which admit free boundaries with constant tangential stress. Starting essentially with the fact that the Kolosov stress combinations $\Theta, \Psi$ can be expressed as

$$\begin{aligned} \Theta &= T_{xx} + T_{yy} = 2(F(z) + \overline{F(z)}) = 4Re(F(z)) \\ \Psi &= T_{xx} - T_{yy} + 2iT_{xy} = -2(\overline{z}F'(z) + G(z)) \end{aligned}$$

where $F, G$ are analytic functions, his problem is reduced to finding functions $F, G$ analytic on a domain $\Omega$, continuous on the boundary $\partial\Omega$ which must be characterised by the vanishing of the normal stress along it, that is

$$\begin{aligned} -T_{xy}dx + T_{xx}dy &= 0 \\ -T_{yy}dx + T_{xy}dy &= 0 \end{aligned}$$

equivalently,

$$2Re(F(z))dz + (\overline{G(z)} + z\overline{F'(z)})d\overline{z} = 0.$$

and by the constancy of $\Theta$, that is

$$Re(F(z)) = \text{constant}$$

Arf carefully axiomatizes the above heuristic scheme ([1], [5]) by adding to it well chosen restrictions on the geometry of the domain as well as on the analytic properties of $F, G$. The picture that emerges is not only theoretically tenable, but also gives a good description of the stress state of an elastic plane region piecewise uniformly loaded at infinity ([2]). As for the fluid dynamical analogy, this is demonstrated on a classical problem of fluid dynamics ([3], [4]).

The future of Arf's work in applied mathematics is not easy to assess. It is nowadays a commonplace observation, paid lip service to by many, taken heed of by few, that despite all absolutist appearances mathematics and mathematicians are not immune to changes in trend. We have many examples of sometime fashionable works and names which have since then dwindled in significance to that much space on our bookshelves. Unexpected comebacks such as that of Arf's work on quadratic forms over fields of characteristic two bear upon the same point. Admittedly it is a fact that Arf's work in applied mathematics, has gone practically unnoticed on an international scale until now. It is also a fact that the most pronounced aspect of modern applied mathematics is the effort that goes into bending rarefield heights of functional analysis into contact with the Gargantuan computational possibilities opened up by electronic devices. From this point of view one may conclude that Arf's work on elasticity constitutes one of the last - as it happens, unrecognized - examples of what might today be called the Heroic Age of Applied Mathematics. However we must remember that, although desirable, efficient computation is not the aim of mathematical sciences. The main thrust of the mathematical élan that dwarves mere fashions is towards phenomenology, in other words, the invariants, or better still in the post-Kleinian terminology, the Geometry. In our opinion, this is a state of affairs in which Arf's work in applied mathematics may well survive.

## References


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*Cem Tezer*