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Motives of some Fano varieties

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Abstract We study the Fano varieties of projective k-planes lying in hypersurfaces and investigate the associated motives.

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1 Introduction

Let $X \subset \mathbb{P}^{n+1}$ be a general smooth hypersurface of degree $d \geq 3$, and assume given a positive integer k satisfying the numerical conditions in main theorem below. Then one can find a smooth projective variety Ω_X of dimension $n - 2k$, parameterizing a family of k-planes in X, such that the essential motivic information about X is encoded in Ω_X via the cylinder correspondence

$$P(X) := \{(c, x) \in \Omega_X \times X \mid x \in \mathbb{P}^k_c\}.$$

Roughly speaking, and up to a normalizing constant, ${}^{\mathrm{T}}P(X) \circ P(X)$ defines a projector on the motive of Ω_X, where by motive, we mean in the sense of Chow motives (with respect to rational equivalence, see [9], page 131). This enables us to decompose the motive of Ω_X in terms of a submotive of X. Our main result is the following:

Theorem 1.1 (Main Theorem). *(i) Let $X \subset \mathbb{P}^{n+1}$ be given above, and assume (k, n, d) satisfy the following:*

$$k = \left[\frac{n+1}{d}\right] \quad and \quad k(n + 2 - k) + 1 - \binom{d+k}{k} \geq 0.$$

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Then there is a motivic decomposition:

$$(\Omega_X, \mathrm{Id}) = (\Omega_X, \tilde{\tau}) \oplus (\Omega_X, \mathrm{Id} - \tilde{\tau}),$$

where $(\Omega_X, \tilde{\tau}, 0) \simeq (X, \tilde{\pi}_n^X, -k)$ *as virtual motives, and* $\tilde{\pi}_X^n$ *is a certain primitive projector associated to the middle dimensional cohomology of* X.

(ii) Let $\sigma = \left({}^{\mathrm{T}}P(X) \circ P(X)\right)_* : \mathrm{CH}^\bullet(\Omega_X) \to \mathrm{CH}^\bullet(\Omega_X)$. *Then there is a short exact sequence:*

$$0 \to (\sigma - m)\mathrm{CH}^{\bullet-k}_{\mathrm{hom}}(\Omega_X; \mathbb{Q}) \to \mathrm{CH}^{\bullet-k}_{\mathrm{hom}}(\Omega_X; \mathbb{Q}) \xrightarrow{\Phi_*} \mathrm{CH}^\bullet_{\mathrm{hom}}(X; \mathbb{Q}) \to 0,$$

where $\Phi_* = P(X)_*$ *and* m *is a nonzero integer defined in* §*4 below. Moreover*

$$\Phi_* : \sigma\left(\mathrm{CH}^{\bullet-k}_{\mathrm{hom}}(\Omega_X; \mathbb{Q})\right) \xrightarrow{\sim} \mathrm{CH}^\bullet_{\mathrm{hom}}(X; \mathbb{Q}),$$

is an isomorphism.

Remarks (i) Part (ii) of the above theorem generalizes the main theorem in [6], where only the case $k = 1$ was considered.

(ii) In the Appendix, we apply our results to Chow-Künneth decompositions in the sense of [9]. For any smooth projective variety Y, which admits a Chow-Künneth decomposition in the sense of Murre, we let π_i^Y be the projector corresponding to $\Delta_Y(2\dim Y - i, i)$, where $\left[\Delta_Y(2\dim Y - i, i)\right] \in H^{2\dim Y - i}(Y, \mathbb{Q}) \otimes H^i(Y, \mathbb{Q})$ induces the identity map on singular cohomology $H^i(Y, \mathbb{Q})$. Murre states a series of conjectures (Conjectures I, II, III, IV in [9]). Our main interest is his Conjecture II, which is a statement about the vanishing of a subset of the projectors $\{\pi_i^Y\}$ on $\mathrm{CH}^\bullet(Y; \mathbb{Q})$. In this Appendix, we generalize this Conjecture II to Bloch's higher Chow groups ([2]), and under the reasonable assumption that (conjecturally!) the projector $\pi_{n-2k}^{\Omega_X}$ can be chosen such that $\pi_{n-2k,*}^{\Omega_X} \circ \tilde{\tau}_* = \tilde{\tau}_* = \tilde{\tau}_* \circ \pi_{n-2k,*}^{\Omega_X}$ on $\mathrm{CH}^\bullet(\Omega_X; \mathbb{Q})$, together with a conjecture of Soulé on the vanishing of certain higher Chow groups of a field, we show that this generalized Conjecture II for Ω_X implies a corresponding (generalized) Conjecture II for X. More precisely,

Theorem 1.2. *Assume the notation and setting in the Main Theorem 1.1. Assume given a Chow-Künneth decomposition of* Ω_X *(in the sense of Murre) such that*

$$\pi_{n-2k,*}^{\Omega_X} \circ \tilde{\tau}_* = \tilde{\tau}_* = \tilde{\tau}_* \circ \pi_{n-2k,*}^{\Omega_X},$$

on $\mathrm{CH}^\bullet(\Omega_X, m; \mathbb{Q})$. *Further, let us assume either that* $m = 0,\ 1,\ 2$ *or a conjecture of Soulé (see Appendix) for* $m \geq 3$. *Then Murre's (generalized) Conjecture II for* Ω_X *implies Murre's (generalized) Conjecture II for* X.

2 Notation

(i) Throughout this paper X will be assumed to be a projective algebraic manifold of dimension n.

(ii) $\mathrm{CH}^r(X)$ is the Chow group of algebraic cycles of codimension r on X, modulo rational equivalence. We put $\mathrm{CH}^\bullet(X;\mathbb{Q}) := \mathrm{CH}^\bullet(X) \otimes \mathbb{Q}$. $\mathrm{CH}^\bullet_{\mathrm{alg}}(X) \subset \mathrm{CH}^\bullet(X)$ is the subgroup of cycles algebraically equivalent to zero, and $\mathrm{CH}^\bullet_{\mathrm{hom}}(X;\mathbb{Q}) \subset \mathrm{CH}^\bullet(X;\mathbb{Q})$ the subspace of nullhomologous cycles.

(iii) The diagonal class of X is denoted by $\Delta_X \in \mathrm{CH}^n(X \times X)$.

(iv) The intersection pairing on $\mathrm{CH}^\bullet(X)$ is denoted by $(\ \bullet\ )_X$.

(v) Let Y be a projective algebraic manifold, and $z \in \mathrm{CH}^r(X \times Y)$. Then $z_* : \mathrm{CH}^\bullet(X) \to \mathrm{CH}^{r-n+\bullet}(Y)$ is given by

$$z_*(\xi) := \mathrm{Pr}_{2,*}\big((\mathrm{Pr}_1^*(\xi) \bullet z)_{X\times Y}\big),$$

and z^* is given by $({}^{\mathrm{T}}z)_*$, where ${}^{\mathrm{T}}z \in \mathrm{CH}^r(Y \times X)$ is the transpose of z.

(vi) If Z is also a projective algebraic manifold, with correspondences $z \in \mathrm{CH}^\bullet(X \times Y)$ and $w \in \mathrm{CH}^\bullet(Y \times Z)$, then:

$$w \circ z := \mathrm{Pr}_{13,*}\big((\mathrm{Pr}_{12}^*(z) \bullet \mathrm{Pr}_{23}^*(w))_{X\times Y\times Z}\big) \in \mathrm{CH}^\bullet(X \times Z).$$

(vii) By a general hypersurface $X \subset \mathbb{P}^{n+1}$ of a given degree, we mean a hypersurface corresponding to a point in a Zariski open subset of the universal family of such hypersurfaces, governed by certain properties (e.g. nonsingularity of X and of Ω_X, etc.).

3 Review of some known results

First some notation: $X \subset \mathbb{P}^{n+1}$ is a general hypersurface of degree $d \geq 3$. We can assume that $X = \mathbb{P}^{n+1} \cap Z$, where $Z \subset \mathbb{P}^{n+2}$ is a general hypersurface of degree d. Fix $k \geq 1$ and for a variety W, let $\Omega_W(k) = \{\mathbb{P}^k\text{'s} \subset W\}$. $\Omega_W \subset \Omega_W(k)$ will denote a given subvariety. We assume that Z is covered by $\mathbb{P}^k$'s, together with this setting:

$$\begin{array}{ccccc}
\Omega_X & & \overset{j_0}{\hookrightarrow} & & \Omega_Z \\
\rho_X \uparrow & & & \rho \nearrow & \uparrow \rho_Z \\
P(X) & \overset{j_1}{\hookrightarrow} & \tilde{X} & \overset{j_2}{\hookrightarrow} & P(Z) \\
\pi_X \downarrow & \pi \swarrow & & & \downarrow \pi_Z \\
X & & \overset{j}{\hookrightarrow} & & Z
\end{array}$$

Figure 1

where: (i) π and π_Z are generically finite to one and *onto* of degree q say.

(ii) $\rho_X : P(X) \to \Omega_X$ and $\rho_Z : P(Z) \to \Omega_Z$ are $\mathbb{P}^k$-bundles.

(iii) $\tilde{X} \stackrel{\mathrm{def'n}}{=} \pi_Z^{-1}(X)$ is smooth.

(iv) $\tilde{\rho} \stackrel{\mathrm{def'n}}{=} \rho|_{\tilde{X}\backslash P(X)} : \tilde{X}\backslash P(X) \to \Omega_Z\backslash\Omega_X$ is a $\mathbb{P}^{k-1}$-bundle.

(v) $\dim X = \dim \tilde{X} = n$, $\dim Z = \dim P(Z) = n+1$, $\dim P(X) = n-k$, $\dim \Omega_X = n-2k$, $\dim \Omega_Z = n-k+1$, and that all varieties in the above diagram are smooth.

Let $H_Z \stackrel{\mathrm{def'n}}{=} \mathbb{P}^{n+1} \cap Z$ be a general hyperplane section of Z, and also set $H_X = H_Z \cap X$.

(vi) $\mu = \pi^{-1}(H_X)$, $\tilde{\mu} = \mu \cap \{\tilde{X}\backslash P(X)\}$, $\mu_Z = \pi_Z^{-1}(H_Z)$, $\mu_X = \pi_X^{-1}(H_X)$.

We will also identify $\{\mu, \tilde{\mu}, \mu_Z, \mu_X\}$ with their respective cohomology classes.

Proposition 3.1 ([7]). *This setting holds in the case where*

$$k = \left[\frac{n+1}{d}\right] \quad and \quad k(n+2-k)+1-\binom{d+k}{k} \geq 0.$$

$\square$

Unless otherwise specified, the above setting, together with the numerical condition in Proposition 3.1 will be assumed throughout the remainder of this paper.

Proposition 3.2 ([7]). *There is an isomorphism*

$$\left\{\bigoplus_{\ell=0}^{k-1} \mathrm{CH}^{\bullet-\ell}(\Omega_Z)\right\} \bigoplus \mathrm{CH}^{\bullet-k}(\Omega_X) \xrightarrow{\sim} \mathrm{CH}^{\bullet}(\tilde{X})$$

given by

$$\left(\sum_{\ell=0}^{k-1} \mu^{\ell} \circ \rho^*\right) + j_{1,*} \circ \rho_X^*.$$

$\square$

We now recall the map $\pi : \tilde{X} \to X$. Then $\pi_* \circ \pi^* = \times q$, and therefore $\pi_* : \mathrm{CH}^{\bullet}(\tilde{X};\mathbb{Q}) \to \mathrm{CH}^{\bullet}(X;\mathbb{Q})$ is surjective. Using the last proposition we note that π_* splits into 2 parts:

(1) $\Phi_* = \pi_* \circ j_{1,*} \circ \rho_X^* = \pi_{X,*} \circ \rho_X^* : \mathrm{CH}^{\bullet-k}(\Omega_X;\mathbb{Q}) \longrightarrow \mathrm{CH}^{\bullet}(X;\mathbb{Q})$ is the cylinder homomorphism.

(2) $\pi_* \circ (\sum_{\ell=0}^{k-1} \mu^\ell \circ \rho^*) : \bigoplus_{\ell=0}^{k-1} \mathrm{CH}^{\bullet-\ell}(\Omega_Z;\mathbb{Q}) \longrightarrow \mathrm{CH}^\bullet(X;\mathbb{Q})$.

<u>We analyze (2)</u>: With the aid of the above diagram, we have:

$$\pi_* \circ (\sum_{\ell=0}^{k-1} \mu^\ell \circ \rho^*) = \pi_* \circ (\sum_{\ell=0}^{k-1} \mu^\ell \circ j_2^* \circ \rho_Z^*)$$

$$= \pi_* \circ j_2^* \circ (\sum_{\ell=0}^{k-1} \mu_Z^\ell \circ \rho_Z^*) = j^* \circ \pi_{Z,*} \circ (\sum_{\ell=0}^{k-1} \mu_Z^\ell \circ \rho_Z^*).$$

It follows from analyzing (2) that the composite below is surjective:

$$\mathrm{CH}^{\bullet-k}(\Omega_X;\mathbb{Q}) \xrightarrow{\Phi_*} \mathrm{CH}^\bullet(X;\mathbb{Q}) \longrightarrow \mathrm{CH}^\bullet(X;\mathbb{Q})/j^*(\mathrm{CH}^\bullet(Z;\mathbb{Q})).$$

To analyze the contribution of $j^*\mathrm{CH}^\bullet(Z;\mathbb{Q})$, we consider a particular choice of Z and the following.

Lemma 3.3 ([6]). *Let $X = V\big(F(z_0,\dots,z_{n+1})\big) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d, and put $Z := V(F+z_{n+2}^d) \subset \mathbb{P}^{n+2}$. Let $j : X \simeq V(z_{n+2}) \cap Z \subset Z$ be the inclusion, $\nu : \mathbb{P}^{n+2} \to \mathbb{P}^{n+1}$ the projection from $[0,\dots,0,1] \in \mathbb{P}^{n+2}$, and $i : X \hookrightarrow \mathbb{P}^{n+1}$ the inclusion. Then with regard to the following (commutative diagram)*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "i"'] & Z \arrow[d, "\nu"] \\ & \mathbb{P}^{n+1} \end{tikzcd}$$

Figure 2

we have

$$dj^* = i^* \circ \nu_*. \qquad \square$$

From now on our choice of Z will be given as in Lemma 3.3, with X of course still assumed general.

Corollary 3.4 ([7]). *$\Phi_* : \mathrm{CH}^{\bullet-k}(\Omega_X;\mathbb{Q}) \longrightarrow \mathrm{CH}^\bullet(X;\mathbb{Q})/\mathbb{Q}\cdot H_X^\bullet$ is surjective.*

Proof.

$$j^*\mathrm{CH}^\bullet(Z;\mathbb{Q}) = i^* \circ \nu_* \mathrm{CH}^\bullet(Z;\mathbb{Q}) = i^*\mathrm{CH}^\bullet(\mathbb{P}^{n+1};\mathbb{Q}) = \mathbb{Q}\cdot H_X^\bullet. \quad \square$$

One can also show that:

Corollary 3.5 ([7]). *(i) $\Phi_* : \mathrm{CH}_{\mathrm{alg}}^{\bullet-k}(\Omega_X) \to \mathrm{CH}_{\mathrm{alg}}^\bullet(X)$ is surjective.*

(ii) $\Phi_* : \mathrm{CH}_{\mathrm{hom}}^{\bullet-k}(\Omega_X;\mathbb{Q}) \to \mathrm{CH}_{\mathrm{hom}}^\bullet(X;\mathbb{Q})$ is surjective. $\square$

4 The kernel of the cylinder map

We would like to compute $\ker \Phi_*$, where Φ_* is given in Corollary 3.4. This has been done in the special case when $k = 1$ in some earlier work ([6]). It is useful to view Φ_* and Φ^* in terms of the correspondences, viz., $\Phi_* = P(X)_*$, and $\Phi^* = \big({}^{\mathrm{T}}P(X)\big)_*$. Now set $\sigma = \Phi^* \circ \Phi_* = \big({}^{\mathrm{T}}P(X) \circ P(X)\big)_*$.

We wish to show that σ satisfies a quadratic relation

$$\sigma \circ (\sigma - m) \equiv 0,$$

where $\equiv$ means equality on $\mathrm{CH}^\bullet(\Omega_X; \mathbb{Q})$ modulo contributions arising from $j^*\mathrm{CH}^\bullet(Z; \mathbb{Q})$ via Φ^*, and where $m = (-1)^k q$ is given by its corresponding multiplication. For this we consider an idea communicated to us by Kapil Paranjape. Namely, the crucial ingredient we need is this:

Proposition 4.1 ([10]). *Let $c \in \Omega_X$ be given. Then*

$$\rho_*\big((P(X) \bullet \mathbb{P}^k_c)_{\tilde{X}}\big) = (-1)^k j_{0,*}(c),$$

where we have identified $\mathbb{P}^k_c$ with $j_{1,*} \circ \rho_X^*(c)$.

Proof. Let G be the Grassmannian of k-planes in $\mathbb{P}^{n+2}$, and let E complete the fiber square below:

$$\begin{array}{ccc} E & \to & U(k+1) \\ \downarrow & & \downarrow \\ \Omega_Z & \hookrightarrow & G, \end{array}$$

i.e. E is the pullback of the universal bundle over G to Ω_Z. Then $\mathbb{P}[E] = P(Z)$. Now recall $\rho_Z : P(Z) \to \Omega_Z$. Then $\rho_Z^*(E)$ lives over $P(Z)$ with tautological bundle $L_Z^* \hookrightarrow \rho_Z^*(E)$. Pulling back to $\tilde{X}$, we define $Q_{k+1}^* = \rho_Z^*(E)\big|_{\tilde{X}}$ and $L^* = L_Z^*\big|_{\tilde{X}}$. Define $Q'^{,*}$ by the s.e.s.:

$$0 \to L^* \to Q_{k+1}^* \to Q'^{,*} \to 0,$$

which dualizes to:

$$0 \to Q' \to Q_{k+1} \xrightarrow{\psi} L \to 0.$$

Let $F = 0$ be the defining equation for $X \subset Z$, and note that F is linear (and homogeneous). Then F defines a section σ_F of Q_{k+1} over $\tilde{X}$ as follows: Let $v \in \mathbb{C}^{k+1} \subset Q_{k+1}^*$ live over a point in $\tilde{X}$. Then $F(v) \in \mathbb{C}$ defines σ_F. It is clearly obvious that σ_F vanishes along $P(X)$ and that $\psi(\sigma_F) = 0$. Note that $\mathrm{rank}(Q') = k$ and that $\sigma_F \in H^0(\tilde{X}, Q')$, hence $c_k(Q') = [P(X)]$. By Whitney,

$$c(Q_{k+1}) = c(Q')c(L) = c(Q')(1 + \xi),$$

where $\xi = c_1(L)$. Hence

$$c(Q') = c(Q_{k+1})(1 + \xi)^{-1} = c(Q_{k+1})\big(1 - \xi + \xi^2 + \cdots + (-1)^n \xi^n\big).$$

Therefore

$$[P(X)] = c_k(Q') = (-1)^k \big(\xi^k - c_1(Q_{k+1})\xi^{k-1} + c_2(Q_{k+1})\xi^{k-2} + \cdots\big).$$

But by functoriality,

$$c_i(Q_{k+1}) = \rho^*\big(c_i(E^*)\big),$$

where we recall $\rho : \tilde{X} \to \Omega_Z$. Observe that for $i > 0$ we can assume that the support of $c_i(E^*) \in \mathrm{CH}^i(\Omega_Z)$ does not meet a given $c \in \Omega_X$. Therefore for such $c \in \Omega_X$,

$$\rho_*\big(\mathbb{P}_c^k \bullet c_i(Q_{k+1}) \bullet \xi^{k-i}\big)_{\tilde{X}} = 0, \text{ for } i > 0.$$

Hence

$$\rho_*\big((P(X) \bullet \mathbb{P}_c^k)_{\tilde{X}}\big) = (-1)^k j_{0,*}(c).$$

In short, the numerical intersection gives $\big(P(X) \bullet \mathbb{P}_c^k\big)_{\tilde{X}} = (-1)^k$. □

Corollary 4.2. *For any* $\xi \in \mathrm{CH}^\bullet(\Omega_X)$*, we have*

$$\rho_{X,*} \circ j_1^* \circ j_{1,*} \circ \rho_X^*(\xi) = (-1)^k \xi.$$

Proof. For a morphism $f : V_1 \to V_2$ of smooth varieties, let $\{f\} \subset V_1 \times V_2$ represent the graph of f. Now put

$$W = \{\rho_X\} \circ {}^{\mathrm{T}}\{j_1\} \circ \{j_1\} \circ {}^{\mathrm{T}}\{\rho_X\}.$$

Then

$$W_* = \rho_{X,*} \circ j_1^* \circ j_{1,*} \circ \rho_X^*,$$

moreover an explicit calculation shows that in $\mathrm{CH}^{n-2k}(\Omega_X \times \Omega_X)$, W is a multiple of the diagonal class Δ_{Ω_X}. By Proposition 4.1, that multiple is precisely $(-1)^k$. □

For $c \in \Omega_X$ put

$$\zeta := \pi^*(\Phi_*(c)) \in \mathrm{CH}^{n-k}(\tilde{X}),$$

and observe that

$$\sigma(c) = \Phi^* \circ \Phi_*(c) = \rho_{X,*} \circ j_1^*(\zeta).$$

By Propositions 3.2 and 4.1, we can write

$$\zeta = \left(\sum_{\ell=0}^{k-1} \mu^\ell \circ \rho^*(\zeta_\ell)\right) + (-1)^k j_{1,*} \circ \rho_X^*\big(\sigma(c)\big)$$

for some $\zeta_\ell \in \mathrm{CH}^{n-k-\ell}(\Omega_Z)$. But modulo $j^*\mathrm{CH}_{k+1}(Z)$,

$$\pi_*\left(\sum_{\ell=0}^{k-1} \mu^\ell \circ \rho^*(\zeta_\ell)\right) \sim_{\mathrm{rat}} 0,$$

and hence if we write $\equiv$ to mean equality modulo $j^*\mathrm{CH}_{k+1}(Z;\mathbb{Q})$ we have

$$q\cdot\Phi_*(c)=\pi_*\circ\pi^*\big(\Phi_*(c)\big)\equiv(-1)^k\Phi_*(\sigma(c)),$$

and

$$\Phi_*\big([\sigma-(-1)^kq](c)\big)\equiv 0.$$

Thus by applying Φ^*, we have

$$\sigma\circ\big([\sigma-m](c)\big)=\Phi^*\circ\Phi_*\big([\sigma-m](c)\big)=0\ \mathrm{modulo}\ \Phi^*\left(j^*\mathrm{CH}_{k+1}(Z;\mathbb{Q})\right).$$

Quite generally, using Corollary 4.2, one can apply the same arguments to arbitrary dimension cycles. More specifically, on $\mathrm{CH}^{\bullet}_{\mathrm{hom}}(\Omega_X;\mathbb{Q})$, as well as on $\mathrm{CH}^{\bullet}(\Omega_X;\mathbb{Q})/\Phi^*\left(j^*\mathrm{CH}^{\bullet+k}(Z;\mathbb{Q})\right)$ one can argue that

$$\sigma\circ\big(\sigma-m\big)=0.$$

We deduce:

Theorem 4.3. *There is a short exact sequence:*

$$0\to(\sigma-m)\mathrm{CH}^{\bullet-k}_{\mathrm{hom}}(\Omega_X;\mathbb{Q})\to\mathrm{CH}^{\bullet-k}_{\mathrm{hom}}(\Omega_X;\mathbb{Q})\xrightarrow{\Phi_*}\mathrm{CH}^{\bullet}_{\mathrm{hom}}(X;\mathbb{Q})\to 0.$$

Moreover

$$\Phi_*:\sigma\left(\mathrm{CH}^{\bullet-k}_{\mathrm{hom}}(\Omega_X;\mathbb{Q})\right)\xrightarrow{\sim}\mathrm{CH}^{\bullet}_{\mathrm{hom}}(X;\mathbb{Q}).$$

□

Next we want to analyze the contribution of $\Phi^*\left(j^*\mathrm{CH}^{k+\bullet}(Z;\mathbb{Q})\right)$ in $\mathrm{CH}^{\bullet}(\Omega_X;\mathbb{Q})$.

Let $H_X^{(j)},\ j=1,2,3,\ldots$ be a general collection of hyperplane sections of X. Observe that

$$\rho_X:\pi_X^{-1}\left(H_X^{(1)}\cap\cdots\cap H_X^{(k)}\right)\xrightarrow{\approx}\Omega_X,$$

is a birational morphism. We note in passing the following.

Proposition 4.4. *Let* $H_{\Omega_X}=\Phi^*\left(H_X^{(1)}\cap\cdots\cap H_X^{(k+1)}\right)\in\mathrm{CH}^1(\Omega_X)$. *Then* H_{Ω_X} *is ample in* Ω_X.

Proof. Let $C\subset\Omega_X$ be any curve.

$$\begin{aligned}(C\bullet H_{\Omega_X})_{\Omega_X}&=\big(C\bullet\Phi^*(H_X^{(1)}\bullet\cdots\bullet H_X^{(k+1)})\big)_{\Omega_X}\\&=\big(\Phi_*(C)\bullet H_X^{(1)}\bullet\cdots\bullet H_X^{(k+1)}\big)_X\\&>0\end{aligned}$$

since $\Phi_*(C)$ is effective. The result now follows from Nakai's criterion. □

Proposition 4.5. $\Phi^*\left(H_X^{(1)}\bullet\cdots\bullet H_X^{(k+i)}\right) = H^i_{\Omega_X} \in \mathrm{CH}^i(\Omega_X)$ *for all* $i\geq 0$, *where* H_{Ω_X} *is given in Proposition 4.4.*

Proof. Put $V_X^{(j)} = H_X^{(1)}\cap\cdots\cap H_X^{(k)}\cap H_X^{(k+j)}$, $j=1,\ldots,i$. It is obvious that $H^i_{\Omega_X} = \left\{\rho_X\left(\pi_X^{-1}(V_X^{(1)}\cap\cdots\cap V_X^{(i)})\right)\right\}\in\mathrm{CH}^i(\Omega_X)$, where $\{(\cdots)\}$ means the class in the Chow group of an intersection operation $(\cdots)$ defined on the level of subvarieties. We then have

$$\begin{aligned}
H^i_{\Omega_X} &= \Phi^*(V_X^{(1)})\bullet\cdots\bullet\Phi^*(V_X^{(i)})\\
&= \left\{\rho_X\left(\pi_X^{-1}(V_X^{(1)}\cap\cdots\cap V_X^{(i)})\right)\right\}\\
&= \left\{\rho_X\left(\pi_X^{-1}(H_X^{(1)}\cap\cdots\cap H_X^{(k)}\cap H_X^{(k+1)}\cap\cdots\cap H_X^{(k+i)})\right)\right\}\\
&= \rho_{X,*}\circ\pi_X^*\left(H_X^{(1)}\bullet\cdots\bullet H_X^{(k+i)}\right)\\
&= \Phi^*\left(H_X^{(1)}\bullet\cdots\bullet H_X^{(k+i)}\right).
\end{aligned}$$

□

Corollary 4.6. $\sigma\circ(\sigma-m)=0$ *on* $\mathrm{CH}^\bullet(\Omega_X;\mathbb{Q})/\mathbb{Q}\cdot H^\bullet_{\Omega_X}$. □

5 Applications to Chow Motives

We work with the aforementioned quadratic relation:

$$\sigma\circ(\sigma-m)=0\quad\text{on }\mathrm{CH}^\bullet(\Omega_X;\mathbb{Q})/\mathbb{Q}\cdot H^\bullet_{\Omega_X},$$

where $\sigma=\Phi^*\circ\Phi_*$. Equivalently, if we replace σ by $\underline{\sigma}:=m^{-1}\sigma$, then we arrive at

$$\underline{\sigma}\circ(\underline{\sigma}-1)=0\quad\text{on }\mathrm{CH}^\bullet(\Omega_X;\mathbb{Q})/\mathbb{Q}\cdot H^\bullet_{\Omega_X}.$$

Note that σ is the map induced by the correspondence ${}^{\mathrm{T}}P(X)\circ P(X)\in\mathrm{CH}^{n-2k}(\Omega_X\times\Omega_X)$, and likewise $\underline{\sigma}$ induced by $\tau:=(m^{-1})\big({}^{\mathrm{T}}P(X)\big)\circ P(X)\in\mathrm{CH}^{n-2k}(\Omega_X\times\Omega_X;\mathbb{Q})$. Furthermore

$$\underline{\sigma}\circ(\underline{\sigma}-1)=0\Rightarrow\underline{\sigma}\circ\underline{\sigma}=\underline{\sigma}.$$

We first show that the correspondence

$$\tau\in\mathrm{CH}^{n-2k}(\Omega_X\times\Omega_X;\mathbb{Q})$$

satisfies

$$\tau\circ(\tau-1)=0\text{ in }\mathrm{CH}^{n-2k}(\Omega_X\times\Omega_X;\mathbb{Q})\Big/\bigoplus_{\ell=k}^{n-k}\mathrm{CH}^{n-k-\ell}(\Omega_X;\mathbb{Q})\otimes H^{\ell-k}_{\Omega_X}.$$

To show this, observe that we can apply the Cartesian product $\Omega_X \times$ to both diagrams in figures 1 and 2. As a formal consequence of our previous results, we arrive at the relation

$$(1\times\sigma)\big((1\times\sigma)-m)\big)(\Delta_{\Omega_X})=0$$

$$\text{in } \mathrm{CH}^{n-2k}(\Omega_X\times\Omega_X;\mathbb{Q})\Big/\bigoplus_{\ell=k}^{n-k}\mathrm{CH}^{n-k-\ell}(\Omega_X;\mathbb{Q})\otimes H^{\ell-k}_{\Omega_X}.$$

But

$$(1\times\sigma)\big((1\times\sigma)-m\cdot 1)\big)(\Delta_{\Omega_X})$$

is precisely

$$\big({}^{\mathrm{T}}P(X)\circ P(X)\big)\circ\big(\big({}^{\mathrm{T}}P(X)\circ P(X)\big)-m\Delta_{\Omega_X}\big)$$

and the aforementioned quadratic relation for τ follows. (Here we use the fact that if W is a smooth projective variety and $\Xi\subset W\times W$ is a correspondence, then $(\Delta_W\times\Xi)_*(\Delta_W)=\Xi$.) Later, we will need to modify τ slightly in order to obtain a quadratic relation on $\mathrm{CH}^{n-2k}(\Omega_X\times\Omega_X;\mathbb{Q})$. Towards this goal, we will introduce in the next section a natural choice of Chow-Künneth decomposition for X.

6 Chow-Künneth Decomposition

For this section only, we will assume that $X\subset\mathbb{P}^{n+1}$ is any given smooth hypersurface.

Let $H^\bullet(X)$ be the singular cohomology of X with $\mathbb{Q}$-coefficients. We have the Künneth decomposition

$$[\Delta_X]\in H^{2n}(X\times X)=\bigoplus_{p+q=2n}H^p(X)\otimes H^q(X).$$

We construct a Chow-Künneth decomposition (in the sense of Murre [9]):

$$\Delta_X=\bigoplus_{p+q=2n}\Delta_X(p,q)\in\mathrm{CH}^n(X\times X;\mathbb{Q}),$$

where

$$[\Delta_X(p,q)]\in H^p(X)\otimes H^q(X),$$

is given as follows. Recall that for $i\neq n$:

$$H^i(X,\mathbb{Q})=\begin{cases}0 & \text{if } i \text{ is odd,}\\ \mathbb{Q}\cdot\big(\mathbb{P}^{n+1-m}\cap X\big)=\mathbb{Q}\cdot H_X^m & \text{if } i=2m \text{ for } 0\le m\le n.\end{cases}$$

For $p+q=2n$, we set

$$\Delta_X(p,q)=\begin{cases}0 & \text{if } p \text{ or } q \text{ is odd,}\\ \frac{1}{\left(H_X^n\right)_X}\left(H_X^\ell\otimes H_X^{n-\ell}\right) & \text{if } (p,q)=(2\ell,2n-2\ell)\neq(n,n),\end{cases}$$

where we observe that $(H_X^n)_X = \deg X$. Then

$$\Delta_X(n,n) = \Delta_X - \sum_{(p,q)\neq(n,n)} \Delta_X(p,q).$$

In $\mathrm{CH}^n(X \times X; \mathbb{Q})$, put

$$\pi_\ell^X = \begin{cases} (\deg X)^{-1} \left(H_X^{n-\ell/2} \times H_X^{\ell/2}\right) & \text{if } \ell \neq n \text{ is even,} \\ 0 & \text{if } \ell \neq n \text{ is odd,} \\ \Delta_X(n,n) & \text{if } \ell = n. \end{cases}$$

We have $\pi_m^X \circ \pi_m^X = \pi_m^X$ and $\pi_m^X \circ \pi_\ell^X = 0$ for $m \neq \ell$. In summary:

Lemma 6.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. The projectors $\{\pi_\ell^X\}$ defined above give a Chow-Künneth decomposition:*

$$\Delta_X = \pi_0^X + \cdots + \pi_{2n}^X.$$

Remarks. Conjecture II by J. Murre ([9], page 149) states that on $\mathrm{CH}^r(X;\mathbb{Q})$, $\pi_{\ell,*}^X = 0$ for $\ell < r$ and for $\ell > 2r$. For $\ell \neq n$, we observe that for dimension reasons alone together with the formula for π_ℓ^X above, that $\pi_{\ell,*}^X = 0$ on $\mathrm{CH}^r(X;\mathbb{Q})$, provided that $\ell \neq 2r$, which is outside the range of Murre's Conjecture II. Thus the only projector to consider is $\pi_{n,*}^X$. But $\ell = n < r$ implies that $\mathrm{CH}^r(X) = 0$ for dimension reasons alone, hence $\pi_{n,*}^X = 0$ for $r < n$. Thus Murre's Conjecture II in this case translates to saying that $\pi_{n,*}^X = 0$ on $\mathrm{CH}^r(X;\mathbb{Q})$ if $2r < n$. However, an affirmative answer to a question of Hartshorne, [4, p142], implies that $\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) = 0$ for $r < n/2$. This further implies Murre's Conjecture II for hypersurfaces (and more generally complete intersections), since for $r < n/2$, $\pi_{n,*}^X \mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) = 0$. We will have more to say about this in the Appendix.

7 Conclusion of the main theorem

Put

$$h_n^X = \begin{cases} (\deg X)^{-1} \left(H_X^{n/2} \times H_X^{n/2}\right) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Put

$$\tilde{\pi}_n^X = \pi_n^X - h_n^X,$$

which we call a primitive projector. Observe that

$$\pi_n^X \circ h_n^X = h_n^X = h_n^X \circ \pi_n^X$$

and hence

$$\tilde{\pi}_n^X \circ h_n^X = h_n^X \circ \tilde{\pi}_n^X = 0.$$

We now want to emphasize that X is now assumed a general hypersurface given as in the setting of Proposition 3.1, with Z given in Lemma 3.3. We need the following result.

Proposition 7.1. $\Phi_* \circ \Phi^* = \times m$ *on* $\mathrm{CH}^\bullet(X;\mathbb{Q})/\mathbb{Q}\cdot H_X^\bullet$.

Proof. We have

$$\Phi_* \circ \Phi^* \circ \Phi_* = \Phi_* \circ \sigma = m\Phi_* \text{ on } \mathrm{CH}^\bullet(X;\mathbb{Q})/\mathbb{Q}\cdot H_X^\bullet.$$

Now use the fact that

$$\Phi_* : \mathrm{CH}^{\bullet-k}(\Omega_X;\mathbb{Q}) \to \mathrm{CH}^\bullet(X;\mathbb{Q})/\mathbb{Q}\cdot H_X^\bullet,$$

is onto. □

By first applying $X\times$ to both diagrams in figures 1 and 2, and using the same reasoning as in §5, we deduce:

Corollary 7.2.

$$P(X) \circ {}^{\mathrm{T}}P(X) - m\Delta_X = 0$$

in

$$\mathrm{CH}^n(X\times X;\mathbb{Q}) \Big/ \bigoplus_{\ell=0}^{n} \mathrm{CH}^{n-\ell}(X;\mathbb{Q})\otimes H_X^\ell.$$

Hence

$$\tilde{\pi}_n^X \circ P(X) \circ {}^{\mathrm{T}}P(X) = m\tilde{\pi}_n^X \text{ in } \mathrm{CH}^n(X\times X;\mathbb{Q}).$$

□

Now put

$$\tilde{\tau} = m^{-1}\big({}^{\mathrm{T}}P(X)\big) \circ \tilde{\pi}_n^X \circ P(X).$$

One easily checks that

$$\tilde{\tau} \circ (\tilde{\tau} - \Delta_{\Omega_X}) = 0 \text{ in } \mathrm{CH}^{n-2k}(\Omega_X \times \Omega_X;\mathbb{Q}),$$

and from this, together with Theorem 4.3, we arrive at the proof of Theorem 1.1 except the proof of the isomorphism of the related motives, which we now show. For the proposition below, we adopt the terminology in [9].

Proposition 7.3. *The motives* $M = (\Omega_X, \tilde{\tau}, 0)$ *and* $N = (X, \tilde{\pi}_n^X, -k)$ *are isomorphic as virtual motives.*

Proof. Define the morphisms

$$\alpha = \frac{1}{m}\,{}^{\mathrm{T}}P(X) \in \mathrm{Corr}^{-k}(X, \Omega_X)$$

and

$$\beta = P(X) \in \mathrm{Corr}^{k}(\Omega_X, X).$$

Then by associativity of correspondences we observe that

$$\tilde{\pi}_n^X \circ \beta \circ \tilde{\tau} \circ \alpha \circ \tilde{\pi}_n^X = \tilde{\pi}_n^X \in \mathrm{Corr}^0(X, X)$$

and

$$\tilde{\tau} \circ \alpha \circ \tilde{\pi}_n^X \circ \beta \circ \tilde{\tau} = \tilde{\tau} \in \mathrm{Corr}^0(\Omega_X, \Omega_X),$$

which establishes the required isomorphism. □

8 Appendix: Murre's conjectures for higher Chow groups

In this section, we will assume the reader has some familiarity with Bloch's higher Chow groups ([2]) $\mathrm{CH}^r(W,m)$, where for our purposes, W is a projective algebraic manifold of dimension n. Further, the reader can consult [9] for the definition of a Bloch-Beilinson filtration $F^{\nu}\mathrm{CH}^r(W;\mathbb{Q})$ on W. Generalizations of the Bloch-Beilinson filtration to the $\mathrm{CH}^r(W,m;\mathbb{Q})$ have been considered by others (e.g. [1], [5], [11]). A generalization of a conjecture of Beilinson says that

$$Gr_F^{\nu}\mathrm{CH}^r(W,m;\mathbb{Q}) \simeq \mathrm{Ext}^{\nu}_{\mathcal{MM}}\big(\mathbf{1}, h^{2r-m-\nu}(W)(r)\big),$$

where $\mathcal{MM}$ is the conjectural category of mixed motives, $\mathbf{1} = \mathrm{Spec}(\mathbb{C})$ is the trivial motive, and $h^{\bullet}(-)$ is motivic cohomology. Implicit in the above formula is an underlying (conjectural) Bloch-Beilinson filtration involving r-steps:

$$\mathrm{CH}^r(W,m;\mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^r \supset \{0\},$$

whose graded pieces factor through the Grothendieck motive. More explicitly, assume given a Chow-Künneth decomposition (or we can work with the weaker assumption of such a decomposition on the level of Grothendieck motives):

$$\Delta_W = \bigoplus_{p+q=2n} \Delta_W(p,q),$$

then

$$Gr_F^{\nu}\mathrm{CH}^r(W,m;\mathbb{Q}) = \Delta_W(2n-2r+\nu+m, 2r-\nu-m)_*\mathrm{CH}^r(W,m;\mathbb{Q}).$$

Again, from the above formula, and for reasons involving weights, one has $F^0 = F^1$ if $m \geq 1$. Recall

$$\pi^W_{\ell,*} := \Delta_W(2n-\ell,\ell)_*.$$

Since we anticipate

$$\Delta_W(2n-2r+\nu+m, 2r-\nu-m)_*\mathrm{CH}^r(W,m;\mathbb{Q}) = 0,$$

for $\nu < 0$ (and if $m > 0$, $\nu \leq 0$) and for $\nu > r$, this translates to

Generalized Murre Conjecture II. $\pi^W_{\ell,*} = 0$ for $\ell > 2r-m$ (and $\ell \geq 2r-m$ if $m > 0$), and for $\ell < r-m$.

We leave it as an exercise for the reader to generalize Murre's remaining conjectures (I, III and IV) to the higher Chow group setting. Before we state our next theorem, we need to recall a conjecture of Soulé:

Conjecture. (Soulé, 1985; see [8]) Let F be a field. Then for $m \geq 2r \geq 2$, $\mathrm{CH}^r(\mathrm{Spec}(F),m;\mathbb{Q}) = 0$. This is an open problem for $r \geq 2$.

We now prove:

Theorem 8.1. *Assume the notation and setting in the Main Theorem 1.1. Assume given a Chow-Künneth decomposition of Ω_X (in the sense of Murre) such that*

$$\pi^{\Omega_X}_{n-2k,*} \circ \tilde{\tau}_* = \tilde{\tau}_* = \tilde{\tau}_* \circ \pi^{\Omega_X}_{n-2k,*},$$

on $\mathrm{CH}^\bullet(\Omega_X, m; \mathbb{Q})$. *Further, let us assume either that* $m = 0,\ 1,\ 2$ *or Soulé's conjecture for* $m \geq 3$. *Then Murre's (generalized) Conjecture II for* $W = \Omega_X$ *implies Murre's (generalized) Conjecture II for* $W = X$.

Proof. By the Main Theorem 1.1,

$$\pi^{\Omega_X}_{n-2k,*} = \tilde{\tau}_* \ + \ \big(\pi^{\Omega_X}_{n-2k,*} - \tilde{\tau}_*\big),$$

is a decomposition into idempotents. Thus

$$\pi^{\Omega_X}_{n-2k,*} = 0 \Rightarrow \tilde{\tau}_* = 0 \Rightarrow \tilde{\pi}^X_{n,*} = 0.$$

We first consider the case $m = 0$. According to the remarks at the end of §6, we need only consider the vanishing of $\tilde{\pi}^X_{n,*}$ on $\mathrm{CH}^r(X;\mathbb{Q})$ when $n > 2r$. Thus it suffices to show that $\pi^{\Omega_X}_{n-2k,*} = 0$ on $\mathrm{CH}^{r-k}(\Omega_X;\mathbb{Q})$ for $n > 2r$. But this is immediate from Murre's (generalized) Conjecture II for Ω_X, since $n - 2k > 2(r-k)$ precisely when $n > 2r$. So now let us assume that $m > 0$. Then we must show that $\pi^X_{\ell,*} = 0$ on $\mathrm{CH}^r(X, m;\mathbb{Q})$ in the ranges $\ell < r - m$ and $\ell \geq 2r - m$. We first introduce

$$h^X_\ell := \begin{cases} 0 & \text{if } \ell \text{ is odd,} \\ H_X^{n-\ell/2} \times H_X^{\ell/2} & \text{if } \ell \text{ is even.} \end{cases}$$

Note that

$$\pi^X_\ell = \begin{cases} h^X_\ell & \text{if } \ell \neq n, \\ \\ \tilde{\pi}^X_n + h^X_n & \text{if } \ell = n. \end{cases}$$

Let $\Delta^m \simeq \mathbb{C}^m$ be the standard algebraic m-simplex as defined in [2]. Any $\xi \in \mathrm{CH}^r(X, m;\mathbb{Q})$ arises from a cycle of codimension r in $X \times \Delta^m$. Consider the product $X \times X \times \Delta^m$. We compute for ℓ even:

$$\begin{aligned} h^X_{\ell,*}(\xi) &= \mathrm{Pr}_{23,*}\big(\mathrm{Pr}^*_{13}(\xi) \bullet \mathrm{Pr}^*_{12}(H_X^{n-\ell/2} \times H_X^{\ell/2})\big) \\ &= \mathrm{Pr}_{23,*}\big\{\big(H_X^{n-\ell/2} \bullet \xi\big) \otimes H_X^{\ell/2}\big\} \\ &\in \mathrm{Pr}_{23,*}\big\{\mathrm{CH}^{n+r-\ell/2}(X, m;\mathbb{Q}) \otimes H_X^{\ell/2}\big\} \\ &\in H_X^{\ell/2} \bullet \lambda^* \mathrm{CH}^{r-\ell/2}(\mathrm{Spec}(\mathbb{C}), m;\mathbb{Q}), \end{aligned}$$

where $\lambda : X \to \mathrm{Spec}(\mathbb{C})$. Note that $\mathrm{CH}^{n+r-\ell/2}(X, m;\mathbb{Q}) = 0$ if $n + r - \ell/2 > n + m$, which is precisely the situation when $\ell < 2(r - m)$. Note that $r < m$ and $\ell < 2(r - m) \Rightarrow \ell < 0$, hence $\dim X = n \Rightarrow H_X^{n-\ell/2} = 0$ for $\ell < 0$ even, and therefore $h^X_{\ell,*}(\xi) = 0$. On the other hand $r \geq m$ and $\ell < r - m \Rightarrow \ell < 2(r - m)$. Thus $\pi^X_{\ell,*} = 0$ for $\ell < r - m$. Next, if $\ell \geq 2r - m$ is even, then $r - \ell/2 \leq [m/2]$, where $[-]$ is the greatest integer function. Thus the vanishing of $h^X_{\ell,*}$ for $\ell \geq 2r-m$

is a consequence of $\mathrm{CH}^{\bullet \leq [m/2]}(\mathrm{Spec}(\mathbb{C}), m; \mathbb{Q}) = 0$, that which is the case for $m = 1,\ 2$, and more generally which is implied by our assumption of Soulé's conjecture. Thus the final step is to show the vanishing of $\tilde{\pi}^X_{n,*}$ in the case where $n < r - m$ and $n \geq 2r - m$. But $n < r - m \Rightarrow r > n + m \Rightarrow \mathrm{CH}^r(X, m) = 0$ for dimension reasons. Thus we are reduced to the case $n \geq 2r - m$. This is equivalent to the statement $n - 2k \geq 2(r - k) - m$ and the vanishing of $\pi^{\Omega_X}_{n-2k,*}$ for $n - 2k \geq 2(r - k) - m$, which is precisely Murre's (generalized) Conjecture II for Ω_X. □

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